Ultrasensitive Two-Mode Interferometry with Single-Mode Number Squeezing

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A major challenge of the phase estimation problem is the engineering of high-intensity entangled probe states. The goal is to significantly enhance above the shot-noise limit the sensitivity of two-mode interferometers. Here we show that this can be achieved by squeezing in input, and then measuring in output, the population fluctuations of a *single* mode. The second input mode can be left as an arbitrary nonvacuum (e.g., a bright coherent) state. This two-mode state belongs to a novel class of particle-entangled states which are *not* spin squeezed. Already a 2.4 db gain above shot noise can be obtained when just a single-particle Fock state is injected into the empty input port of a classical interferometer configuration. Higher gains, up to the Heisenberg limit, can be reached with squeezed states of a larger number of particles. We finally study the robustness of this protocol with respect to detection noise.

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Optical and atomic interferometers are among the most sensitive devices for metrology and weak forces detection [1,2]. Most interferometers can be mapped on the linear Mach-Zehnder (MZ) configuration, Fig. 1(a), where the phase shift is equally imprinted on each particle entering the device. The sensitivity of the phase estimation crucially depends on the nature of the particles quantum state. Current interferometer technology exploits high-intensity classical fields, reaching a phase uncertainty scaling as the inverse square root of the average total number of particles. This bound is known as the shot-noise (or standard quantum) limit. The phase uncertainty can be further reduced, down to the fundamental quantum level (the so-called Heisenberg limit), by engineering proper particle-entangled input states [3,4]. This is a groundbreaking prediction of quantum mechanics which is under intense experimental [5–12] and theoretical [13] investigation. However, in several technological applications, such protocol needs nonclassical sources of high intensity. To achieve this goal, current atomic and optical schemes exploit spin-squeezing techniques [14]. In particular, on the optical side [2,12], this is achieved with a high-intensity coherent state in one input port and a low intensity squeezed-vacuum state in the other input, as first proposed by Caves [15].

In this manuscript we discuss an alternative scheme to reach a sub-shot-noise phase uncertainty with high-intensity states. We study

$$\hat{\rho} = \hat{\rho}_a \otimes |N\rangle_b\langle N|, \tag{1}$$

where $\hat{\rho}_a$ is an *arbitrary* (for instance, coherent or thermal) nonvacuum state in mode a and $|N\rangle_b$ is a Fock state of N particles in mode b. We show that Eq. (1) is particle entangled and, when used as input of a MZ interferometer, Fig. 1(a), can provide a sub-shot-noise phase uncertainty down to the Heisenberg limit. We also demonstrate that Eq. (1) belongs to a novel class of states which are not spin squeezed and do not require a precise phase relation between the two modes (in contrast, for instance, to the case of Ref. [15]). Finally, our main results can be generalized by replacing the Fock state in Eq. (1) with a number-squeezed state [16].

The state (1) can be experimentally created with a large number of particles: the input mode a can be a high-intensity coherent state, while Fock states of various particle numbers are nowadays experimentally available. In the context of phase estimation, Fock states have been created by parametric down-conversion with photons [9–11] (see also Ref. [17]) and by spin-changing collisions with Bose-Einstein condensates [8] (see also Ref. [18]). With cold atoms, single-mode Fock states may be provided by nondestructive atom-light interaction [19].

The ultimate phase uncertainty achievable in a MZ interferometer with a generic input state $\hat{\rho}_{\rm inp}$ is given by the quantum Cramer-Rao (QCR) bound [3,4,20–22], $\Delta\theta_{\rm QCR} = 1/\sqrt{mF_Q[\hat{\rho}_{\rm inp}, \hat{J}_y]}$. Here $F_Q[\hat{\rho}_{\rm inp}, \hat{J}_y]$ is the quantum Fisher information, m is the number of measurements, the operators $\hat{J}_x = (\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})/2$, $\hat{J}_y = (\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})/2i$, and $\hat{J}_z = (\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})/2$, commute with the total number of particles operator $\hat{n} = \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}$ and satisfy the commutation relations for the Lie algebra of SU(2), where $\hat{a}$ and $\hat{b}$ ($\hat{a}^\dagger$ and $\hat{b}^\dagger$) are bosonic mode annihilation (creation) operators. The unitary MZ transformation is $e^{-i\theta\hat{J}_y}$ [23], where θ is the unknown phase shift. A direct calculation with Eq. (1) gives $F_Q[\hat{\rho}, \hat{J}_y] = 2Nn_a + N + n_a$ [22], and thus

$$\Delta\theta_{\rm QCR} = \frac{1}{\sqrt{m}\sqrt{2Nn_a + N + n_a}}, \tag{2}$$

where $n_a = {\rm Tr}[\hat{a}^\dagger\hat{a}\hat{\rho}_a]$ is the average number of particles in $\hat{\rho}_a$. It is interesting to discuss a few limit cases. If one of the two modes is empty ($N = 0$ or $n_a = 0$), the phase uncertainty Eq. (2) is, as expected, given by the shot-noise limit,

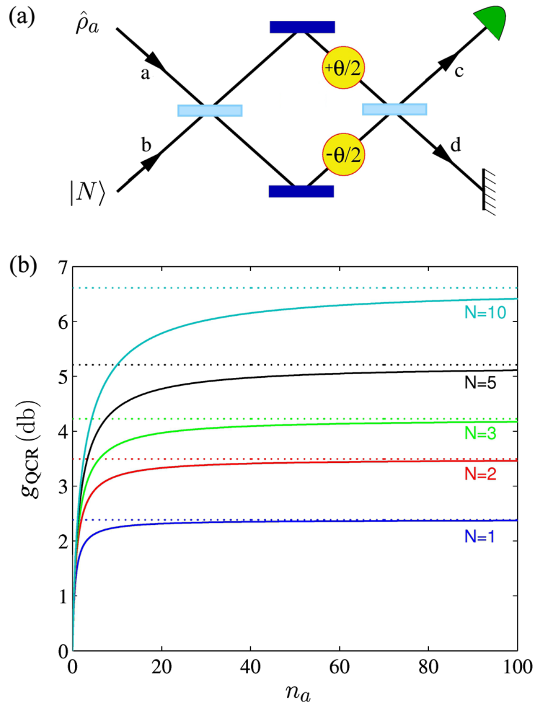

FIG. 1 (color online). (a) Scheme of a Mach-Zehnder interferometer with input state given by Eq. (1) and number of particles detection in a single output port. The parameter θ is the relative phase shift among the two arms of the interferometer. (b) Phase sensitivity gain $g_{\rm QCR} \equiv -10\log_{10}(\Delta\theta_{\rm QCR}/\Delta\theta_{\rm sn})$ of Eq. (2) with respect to the shot-noise limit $\Delta\theta_{\rm sn} = 1/\sqrt{m}\sqrt{n_a + N}$. Different lines refer to different values of N. The horizontal lines [corresponding to $\Delta\theta_{\rm QCR}/\Delta\theta_{\rm sn} = 1/\sqrt{2N+1}$, Eq. (3)] are saturated in the limit $n_a \gg N$.

$\Delta\theta_{\rm sn} = 1/\sqrt{m\langle\hat{n}\rangle}$, where $\langle\hat{n}\rangle = N + n_a$. We obtain a sub-shot-noise uncertainty as soon as both the input modes are not empty ($Nn_a > 0$). An important limit is when $\hat{\rho}_a$ contains most of the particles (e.g., it is a large intensity coherent state). In this case the phase uncertainty is

$$\Delta\theta_{\rm QCR} = \frac{1}{\sqrt{2N+1}}\frac{1}{\sqrt{m\langle\hat{n}\rangle}} \quad \text{for } n_a \gg N, \tag{3}$$

where $\langle\hat{n}\rangle = (n_a + N) \approx n_a$ is the average number of particles. Equation (3) is below $\Delta\theta_{\rm sn}$ by a factor $\sqrt{2N+1}$. Notice that already with a single particle ($N = 1$) in mode b [24] it is possible to overcome the shot noise by 2.4 db, while $\sim$7 db can be obtained with $N \sim 10$, see Fig. 1(b). For a fixed average number of particles $\langle\hat{n}\rangle = n_a + N \gg 1$, the lowest phase uncertainty is obtained for $N = n_a = \langle\hat{n}\rangle/2$ and Eq. (2) predicts

$$\Delta\theta_{\rm QCR} = \frac{\sqrt{2}}{\langle\hat{n}\rangle\sqrt{m}} \quad \text{for } n_a = N \gg 1. \tag{4}$$

We thus recover a phase uncertainty at the Heisenberg limit [4] (modulo a factor $\sqrt{2}$). As already emphasized, Eqs. (2)–(4) do not depend on the specific properties of the input state $\hat{\rho}_a$. In particular, if $\hat{\rho}_a = |N\rangle_a\langle N|$, Eq. (1) reduces to the twin-Fock state first studied in Ref. [25] and recently experimentally investigated in Refs. [8–11].

So far, we have restricted our discussion to the QCR uncertainty. It is well known that this bound can be saturated by optimal generalized measurements [21]. However, in current experiments, the phase shift is generally estimated by measuring the number of particles in one or both of the output ports of the interferometer. We first show that, when measuring the number of particles in a *single* output port of the MZ, Fig. 1(a), it is possible to obtain a phase estimation saturating the prediction of the QCR at optimal phase values. In this case, since the other output is unused, our interferometer configuration is suitable for power recycling. When considering the case $n_a \gg N$ [thus achieving the phase uncertainty predicted by Eq. (3)], the phase estimation requires the measurement of the small intensities. Detectors for a small number of particles are available with high efficiency. We will later show that, when measuring the number of particles at *both* output ports of the MZ, we can obtain a phase uncertainty saturating the QCR bound at any value of the phase shift.

The probabilities of output measurements, given by the phase shift θ, are the relevant ingredient for phase estimation. For a MZ with generic input $\hat{\rho}_{\rm inp}$, the probability to measure N_c and N_d particles at the two output ports is given by $P(N_c, N_d|\theta) = \langle N_c, N_d|e^{-i\theta\hat{J}_y}\hat{\rho}_{\rm inp}e^{i\theta\hat{J}_y}|N_c, N_d\rangle$. Because of the conservation of particle number, for input state Eq. (1) this simplifies to $P(N_c, N_d|\theta) = \rho_{N_c+N_d-N}\langle N_c, N_d|e^{-i\theta\hat{J}_y}|N, N_c + N_d - N\rangle^2$, where the rotation matrix element is real and $\rho_n = \langle n|\hat{\rho}_a|n\rangle$. In the case of single detection mode, the probability to measure N_c particles at the output port c is $P(N_c|\theta) = \sum_{N_d} P(N_c, N_d|\theta)$. For the input state Eq. (1) we find $P(N_c|\theta) = \sum_{n\geq n_0}^{+\infty} \rho_n\langle N_c, N + n - N_c|e^{-i\theta\hat{J}_y}|N, n\rangle^2$ where $n_0 = \max\{0, N_c - N\}$. For an explicit expression of $P(N_c, N_d|\theta)$ and $P(N_c|\theta)$ in terms of Jacobi polynomials, see Ref. [22]. Given these conditional probabilities, the phase uncertainty can be calculated as the Cramer-Rao (CR) bound for unbiased estimators [22,26]. In general, there is no guarantee that an unbiased estimator saturating this bound exists for an arbitrary small number of measurements m [27]. However, asymptotically in m, it is known [26] that the maximum likelihood estimator is unbiased and its variance equals the CR bound. In the following we will thus consider m to be sufficiently large.

In the case of a single output measurement, the CR bound is $\Delta\theta_{\rm CR} = 1/\sqrt{mF_c[\hat{\rho}, \theta]}$, where

$$F_c[\hat{\rho}, \theta] = \sum_{N_c=0}^{+\infty} \frac{1}{P(N_c|\theta)} \left(\frac{dP(N_c|\theta)}{d\theta} \right)^2 \qquad (5)$$

is the Fisher information. In the ideal noiseless case, the highest value of the Fisher information, Eq. (5), is obtained at $\theta = 0$. The explicit calculation at $\theta \approx 0$, gives $F_c[\hat{\rho}, \theta] = 2n_a N + n_a + N + O(\theta^2)$ [22]: the Fisher information saturates the quantum Fisher information, $F_c[\hat{\rho}, \theta = 0] = F_Q[\hat{\rho}, \hat{J}_y]$, and the CR bound saturates Eq. (2). A plot of $\Delta\theta_{\rm CR}$ as a function of θ is shown in Fig. 2 for different values of N. The uncertainty $\Delta\theta_{\rm CR}$ strongly depends on θ and sub-shot noise is obtained in a wide phase interval which slowly shrinks by increasing N. Finite fluctuations in the phase uncertainty are reduced by increasing n_a and are a consequence of the oscillating properties of the conditional probabilities.

The value of the Fisher information at $\theta = 0$ is undetermined (0/0) although the limit exists due to a perfect compensation between numerator and denominator of Eq. (5) [22]. Therefore, this limit case is extremely fragile in the presence of noise and decoherence as, for instance, detection noise. We simulate detection noise by replacing the ideal probabilities $P(N_c|\theta)$ with $\tilde{P}(N_c|\theta) = \sum_{\tilde{N}_c} P(N_c|\tilde{N}_c) P(\tilde{N}_c|\theta)$, where $P(N_c|\tilde{N}_c)$ is the probability to obtain the result N_c when $\tilde{N}_c$ particles truly hit the detector [28]. This can be typically modeled as an unbiased Gaussian, $P(N_c|\tilde{N}_c) \approx e^{-(N_c - \tilde{N}_c)^2/2\sigma^2}$, where σ vanishes in the ideal limit. We assume, for simplicity, that the detection noise parameter σ does not depend on N_c and $\tilde{N}_c$. The CR bound, taking into account this noise, becomes $\Delta\theta_{\rm CR} = 1/\sqrt{m\tilde{F}_c[\hat{\rho}, \theta]}$, where $\tilde{F}_c[\hat{\rho}, \theta]$ is obtained from Eq. (5) by replacing $P(N_c|\theta)$ with $\tilde{P}(N_c|\theta)$. Notably, $\tilde{F}_c[\hat{\rho}, \theta = 0] = 0$ when $\sigma > 0$: the point $\theta = 0$, which was the optimal phase value in the ideal case, becomes the worst working point even in the presence of infinitesimally small detection noise. An analogous effect has been experimentally observed in Ref. [8]. As shown in Fig. 2, the optimal phase value depends on N and σ. Interestingly, due to the properties of $\tilde{F}_c[\hat{\rho}, \theta]$, our scheme proves the most robust in the two extreme limits $N \approx 1$ and $N \gg 1$. In the case $N \approx 1$, sub-shot noise is obtained for $\sigma \lesssim 3$. In the case $N \gg 1$, numerical calculations suggest that sub-shot noise is obtained for $\sigma \lesssim 0.6\sqrt{N}$.

Furthermore, it is interesting to calculate the phase sensitivity of the MZ configuration where both output ports are monitored. The CR bound is now given by $\Delta\theta_{\rm CR} = 1/\sqrt{mF_{cd}[\hat{\rho}, \theta]}$, where the Fisher information is

$$F_{cd}[\hat{\rho}, \theta] = \sum_{N_c, N_d} \frac{1}{P(N_c, N_d|\theta)} \left(\frac{dP(N_c, N_d|\theta)}{d\theta} \right)^2. \qquad (6)$$

By using a Cauchy-Schwarz inequality, it is possible to prove [22] that $F_{cd}[\hat{\rho}, \theta] \geq F_c[\hat{\rho}, \theta]$, for any θ and any $\hat{\rho}$. In our case, since $F_c[\hat{\rho}, \theta = 0] = F_Q[\hat{\rho}, \hat{J}_y]$ and $F_{cd}[\hat{\rho}, \theta] \leq F_Q[\hat{\rho}, \hat{J}_y]$, we also have $F_{cd}[\hat{\rho}, \theta = 0] = F_Q[\hat{\rho}, \hat{J}_y]$. It is also possible to see that $F_{cd}[\hat{\rho}, \theta]$ does not depend on θ [22]. Measuring the number of particles at both output ports thus provides a Cramer-Rao phase uncertainty $\Delta\theta_{\rm CR} = 1/\sqrt{mF_{cd}[\hat{\rho}, \theta]}$ saturating the QCRB at *any* value of the phase shift. This is confirmed by numerical calculations, as shown in Fig. 2.

In the presence of a superselection rule forbidding coherences between different total number of particles [29], sub-shot-noise phase uncertainty in a linear interferometer is directly related to the entanglement properties of the input state [3,4]. The definition and presence of entanglement depends on a specific, chosen, partition of the Hilbert space. In interferometry, states which are only

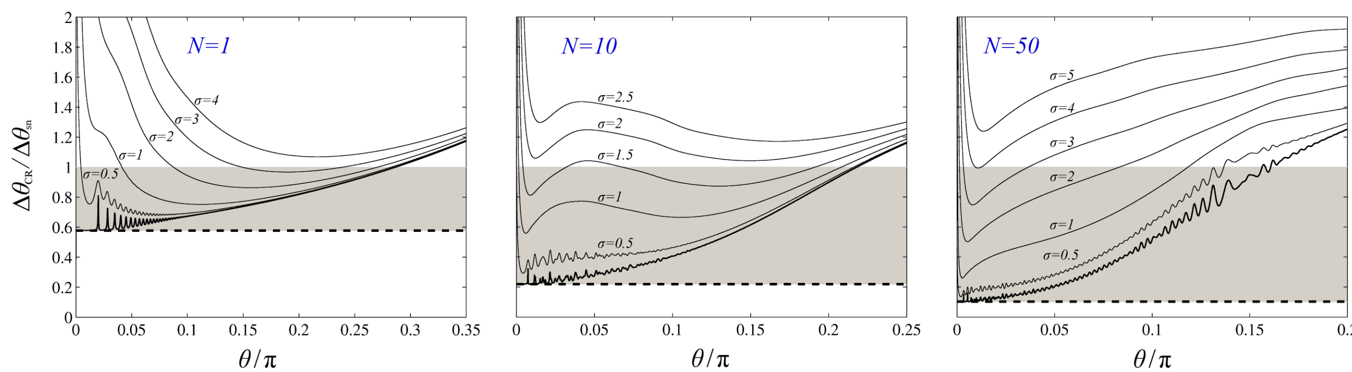

FIG. 2 (color online). Mach-Zehnder phase uncertainty as a function of θ for the input states Eq. (1). Here $\hat{\rho}_a$ is a coherent state ($\rho_n = \langle n|\hat{\rho}_a|n\rangle = n_a^n e^{-n_a}/n!$) with $n_a = 1000$ and different panels refer to different values of N. Thick solid line is the noiseless case obtained by measuring the number of particles in the output port c. In this case $\Delta\theta_{\rm CR}/\Delta\theta_{\rm sn} = \sqrt{\langle\hat{n}\rangle/F_c[\hat{\rho}, \theta]}$ where $F_c[\hat{\rho}, \theta]$ is given by Eq. (5) and $\langle\hat{n}\rangle = n_a + N$. Thin solid lines are obtained by including detection noise (parametrized by σ), $\Delta\theta_{\rm CR}/\Delta\theta_{\rm sn} = \sqrt{\langle\hat{n}\rangle/\tilde{F}_c[\hat{\rho}, \theta]}$. The thick horizontal dashed line is obtained by the noiseless measurement of the number of particles at both the output ports. In this case $\Delta\theta_{\rm CR}/\Delta\theta_{\rm sn} = \sqrt{\langle\hat{n}\rangle/F_{cd}[\hat{\rho}, \theta]}$ where $F_{cd}[\hat{\rho}, \theta]$ is given by Eq. (6). The shaded region highlights the attainable sub-shot-noise phase uncertainty, which is limited from below by the QCR bound, Eq. (2).

classically correlated in the particles provide, at best, the shot-noise limit [3,4]. Notice that the linear interferometric transformations, as the MZ case, are local in the particles and, therefore, no particle entanglement is created by the device. The inequality $F_Q[\hat{\rho}_{\rm inp}, \hat{J}_y] > \langle \hat{n} \rangle$ recognizes the generic two-mode input $\hat{\rho}_{\rm inp}$, of average $\langle \hat{n} \rangle$ particles, as particle entangled and useful for sub-shot-noise interferometry. A different description of the system is provided by mode entanglement. The disadvantage of this point of view is that mode entanglement may be modified by linear SU(2) operations, such as a beam splitter, which makes it difficult to quantify as a resource. Moreover, mode entanglement is, in general, not a good criterion to single out states allowing for phase estimation sensitivities higher than the classical shot noise. An example is Eq. (1), which is mode separable but particle entangled and provides a sub-shot-noise phase uncertainty. To demonstrate this point, we notice, by following [4], that

$$F_Q[\hat{\rho}_{\rm inp}, \hat{J}_y] \geq \langle \hat{n} \rangle / \xi^2_{\rm sm}, \tag{7}$$

where

$$\xi^2_{\rm sm} \equiv \langle \hat{n} \rangle (\Delta[\hat{n} - \hat{J}_z])^2 / \langle \hat{J}_x \rangle^2. \tag{8}$$

It is possible to see that coherent single-mode number squeezing implies $\xi^2_{\rm sm} < 1$. Indeed, let us consider a generic mode-separable input state $\hat{\rho}_{\rm inp} = \hat{\rho}_a \otimes \hat{\rho}_b$ with $n_a \gg n_b$ (such that $\langle \hat{n} \rangle = n_a + n_b \approx n_a$) and number coherences in both modes, $\langle \hat{a} \rangle \sim \sqrt{n_a}$ ($\langle \hat{b} \rangle \sim \sqrt{n_b}$): in this case, $\xi^2_{\rm sm} < 1$ as soon as $\hat{\rho}_b$ is number squeezed [16]. According to Eq. (7), $\xi^2_{\rm sm} < 1$ implies $F_Q[\hat{\rho}_{\rm inp}, \hat{J}_y] > \langle \hat{n} \rangle$, which guarantees that $\hat{\rho}_{\rm inp}$ is particle entangled [4]. For the extreme single-mode number squeezing given by Eq. (1), the quantity $\xi^2_{\rm sm}$ is undetermined (0/0). However, the direct calculation of $F_Q[\hat{\rho}, \hat{J}_y]$ (see Ref. [22]) shows that $F_Q[\hat{\rho}, \hat{J}_y] > N + n_a$, and therefore $\hat{\rho}$ is particle entangled, as soon as $N n_a > 0$; i.e., none of the two modes is in a vacuum state.

It should be noticed that Eq. (8) resembles the familiar spin-squeezing parameter $\xi^2 = \langle \hat{n} \rangle (\Delta \hat{J}_u)^2 / (\langle \hat{J}_v \rangle^2 + \langle \hat{J}_w \rangle^2)$ [14], where u, v, and w are three mutually orthogonal unit vectors. Indeed, $\xi^2_{\rm sm} = \xi^2$ for $u = z$ and a proper choice of v and w, when there are no fluctuations in the total number of particles. Analogously to the spin-squeezing parameter, Eq. (8) is metrologically relevant: states satisfying $\xi^2_{\rm sm} < 1$ reach a sub-shot-noise phase uncertainty at $\theta = 0$, as it can be shown by an error propagation analysis [22]. In general, number squeezing in a single mode does not necessarily imply spin squeezing. We show this by calculating ξ^2 for the state (1). It is straightforward to obtain that $\xi^2 = \frac{\langle \hat{n} \rangle \langle \hat{J}_x^2 \rangle}{\langle \hat{J}_z \rangle^2} + \frac{\langle \hat{n} \rangle (\Delta \hat{J}_z)^2}{\langle \hat{J}_z \rangle^2} d(u, v, w)$, where $d(u, v, w) \geq 0$ is a positive function of the angles defining u, v, w with respect to the x, y, z axis. The spin moments calculated for Eq. (1) are $\langle \hat{J}_{x,y} \rangle = 0$, $\langle \hat{J}^2_{x,y} \rangle = (2Nn_a + N + n_a)/4$, and $\langle \hat{J}_z \rangle = (n_a - N)/2$. Therefore, $\xi^2 \geq \frac{(n_a + N)(2Nn_a + N + n_a)}{(n_a - N)^2} \geq 1$, reaching its minimum ($\xi^2 = 1$) when one of the input mode is in the vacuum state ($N = 0$ or $n_a = 0$). This demonstrates that (1) is *not* spin squeezed for any choice of u, v, w.

In conclusion, we have shown that squeezing the population fluctuations in a single input mode of a Mach-Zehnder interferometer, while leaving the other input in an arbitrary nonvacuum state, creates a particle entangled state useful to reach sub-shot-noise phase uncertainties down to the Heisenberg limit. Since our scheme uses a classical input mode, it is particularly feasible to create particle-entangled input states of large intensity, which are an essential ingredient in several interferometric technological applications as atomic clocks and gravitational wave detectors.

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