# Hierarchies of Multipartite Entanglement

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We derive hierarchies of separability criteria that identify the different degrees of entanglement ranging from bipartite to genuine multipartite in mixed quantum states of arbitrary size.

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Quantum coherence is deemed responsible for a large variety of features, ranging from fundamental physical effects such as superfluidity, via a broad range of counter-intuitive interference and correlation phenomena with potential implications in the realm of quantum information technologies [1] to transport processes [2] even at the mesoscopic scale on the border between physics, chemistry, and biology [3]. A central coherence property of, e.g., a photonic wave packet is its coherence length, but the extension of such a concept to composite quantum systems is by no means straightforward.

Entanglement theory promises an accurate characterization of coherence properties in multipartite quantum systems in terms of $k$-partite entanglement, i.e., the minimum number $k$ of entangled components necessary to describe an $n$-partite system (in literature also referred to as depth of entanglement [4] or $k$ producibility [5]). The definition of these concepts [as given in Eq. (1) below] is rather elementary, but, due to its nonconstructive nature, the identification of $k$-partite entanglement in given mixed quantum states is a largely open problem: up to now the theory of bipartite entanglement (i.e., $k = 2$) has been developed fairly well [6], and there has been substantial progress in the identification of genuine $n$-partite entanglement (i.e., $k = n$) [7–9]. On the scales in between, for $n > k > 2$, however, only punctual knowledge, typically for states of specific type or size, is currently available [4,5,10].

The ability to probe these scales in between is highly desirable for various reasons: while it is well established that quantum computations with pure states necessarily require a large amount of entanglement in order to perform beyond the classically achievable [11], the situation is not as evident for mixed states as they would occur in realistic implementations, since also mixed separable states can lead to improved computational power [12,13]. The possibility to identify entanglement properties in a more fine-grained version than currently possible for the mixed case would certainly help the understanding of which specific features of multipartite quantum states are really necessary for the appraised quantum speed-up.

In precision interferometry, the full enhancement of precision based on $n$ particles can be obtained only for a genuinely $n$-partite entangled state [14]. Entanglement between fewer components will result in a precision closer to the achievable with $n$ independent particles: identifying the largest $k$-partite entanglement (for $k \leq n$) that can be realized at given experimental conditions provides therefore very rigorous limitations to the achievable precision.

Similarly, such an assessment permits to estimate the number of nodes over which coherence in a computational network has been achieved [15]. Fast excitation transport through molecular or spin networks has been shown to be associated with quantum coherence between an intermediate number of nodes [16], and such coherence can be identified through $k$-partite entanglement after projection onto the single-excitation subspace [17]. This provides a very accurate characterization of the spatial extent over which a multipartite system displays quantum mechanical features, and the environmentally induced degradation of coherence can then be followed to monitor the emergence of classicality in a rather detailed fashion.

Our goal in the present contribution is, therefore, to provide for any system size $n$ a full hierarchy of separability criteria to characterize multipartite entanglement: the criterion at the top of each hierarchy identifies genuine $n$-partite entanglement, followed by criteria that are positive only for states with at least $k$-partite entanglement for $k$ ranging from $n - 1$ to 2.

Before introducing our framework, let us review briefly the necessary formal background. A pure state $|\Psi_{n,n}\rangle$ of an $n$-partite quantum system is considered $n$-partite entangled if there is no separation of the subsystems into two groups, such that $|\Psi_{n,n}\rangle$ could be described as the tensor product of states of these two groups. Analogously, an $n$-partite state $|\Psi_{k,n}\rangle$ is considered $k$-partite entangled if it cannot be described without an at least $k$-partite entangled contribution. If a state is not at least bipartite entangled, then it is separable. For pure states, definition and identification of $k$-partite entanglement is rather straightforward, but the situation changes drastically for mixed states: a mixed $n$-partite state is considered $k$-partite entangled if it cannot be expressed as a statistical mixture

$$\sum_i \sum_{j=1}^{k-1} p_{ij} |\Psi_{j,n}^{(i)}\rangle\langle\Psi_{j,n}^{(i)}| \neq \varrho_{k,n}, \tag{1}$$

 

of at most $(k-1)$-partite entangled states with $p_{ij} \geq 0$ [5]. This leads to a rather intricate structure of multipartite entanglement as sketched in Fig. 1.

The task of our present hierarchies of separability criteria is to provide a potentially accurate identification of $k$-partite entanglement in mixed states. We first start out describing the underlying idea in rather general terms, followed by a specific realization that satisfies all of the desired properties. What we aim at is a set of functions

$$\tau_{k,n}(\varrho) = f(\varrho) - \sum_{i=1}^{n/2} a_i^{(k,n)} \sum_j f_{ij}(\varrho) \qquad (2)$$

defined in terms of functions $f$ and $f_{ij}$ where the index $j$ labels all inequivalent bipartitions [18] of the $n$-partite system in an $i$-partite and an $(n-i)$-partite component, referred to as $i$-bipartitions in the following. Furthermore, the functions $f$ and $f_{ij}$, and the scalar weight factors $a_i^{(k,n)}$ need to satisfy the following conditions

(i) $\tau_{k,n}(\varrho)$ is convex, i.e., $\sum_i p_i \tau(\varrho_i) \geq \tau(\sum_i p_i \varrho_i)$,
(ii) $f(\varrho) \geq 0$ and $f_{ij}(\varrho) \geq 0$, $\forall i, j, \varrho \geq 0$,
(iii) $f(\Psi) = f_{ij}(\Psi)$ if $|\Psi\rangle$ is biseparable with respect to the $j$th $i$-bipartition,
(iv) $a_i^{(k,n)} \geq 0$, $\forall i, k, n$.

Convexity of $\tau_{k,n}$ allows us to restrict the following discussion to pure states: if $\tau_{k,n}$ is nonpositive for all pure states with less than $k$-partite entanglement, condition (i) entails that a positive value of $\tau_{k,n}$ identifies (at least) $k$-partite entanglement in mixed states. What remains to be done is to tailor the prefactors $a_i^{(k,n)}$ in order for $\tau_{k,n}$ to have the desired properties for pure states. Since $f_{ij}(\Psi)$ coincides with $f(\Psi)$ if $|\Psi\rangle$ is separable with respect to the $j$th $i$-bipartition, it is sufficient to characterize the separability properties of pure $k$-partite entangled $n$-partite states, and choose the weights $a_i^{(k,n)}$ such that $\sum_{ij|bs} a_i^{(k,n)} f_{ij}(\Psi_{k'n}) \geq f(\Psi_{k'n})$ for any $k'$-partite entangled state with $k' < k$, where the sum runs over all bipartitions with respect to which $|\Psi_{k'n}\rangle$ is separable. Because of the positivity of $f_{ij}$ and $a_i^{(k,n)}$ this directly implies that $\tau_{k,n}$ is nonpositive for all states which are not at least $k$-partite entangled.

As indicated in Fig. 2, pure states with only a small entangled component are biseparable with respect to many bipartitions, so that many components $f_{ij}(\Psi)$ coincide with $f(\Psi)$ and the weights can be chosen comparatively small. Choosing the weight factors $a_i^{(k,n)}$ increasing with $k$ will thus allow us to arrive at the desired hierarchies.

At the bottom of the hierarchies lies $\tau_{2,n}$ which has to be nonpositive for all completely separable states $|\Psi_{1,n}\rangle$. Since $|\Psi_{1,n}\rangle$ is separable with respect to any bipartition, we have $f(\Psi_{1,n}) = f_{ij}(\Psi_{1,n})$ $\forall i, j$ due to (ii). Any choice satisfying $\sum_i a_i^{(2,n)} = 1$ will therefore result in $\tau_{2,n}(\Psi_{1,n}) = 0$ for any completely separable state $|\Psi_{1,n}\rangle$. In order to proceed we need to tailor the $a_i^{(3,n)}$ such that $\tau_{3,n}$ is nonpositive for all pure states that contain less than tripartite entanglement. As depicted in Fig. 2 with the exemplary case of $n = 5$, any pure state $|\Psi_{2,n}\rangle$ (for $n \neq 4$) is separable with respect to at least $m$ 2-bipartitions, where $m$ is the largest integer $\leq n/2$ [19]. Accordingly, $a_i^{(3,n)} = \delta_{i2}/m$ is a valid choice for $\tau_{3,n}$.

Typically, there is not a unique choice for the weights $a_i^{(k,n)}$, and the resulting freedom can be used to optimize the functions $\tau_{k,n}$ for specific quantum states. As a rough rule of thumb we found that for states with highly mixed reduced density matrices choices with large weight factors $a_i^{(k,n)}$ for $i \simeq n/2$ and small or vanishing ones for $i \ll n/2$ yield strong criteria. For example for odd $n > 5$, $a_i^{(3,n)} = \delta_{i3}/m$ and $a_i^{(3,n)} = \delta_{i2}/m$ are both valid choices to define $\tau_{3,n}$, but we found the former to result in a stronger

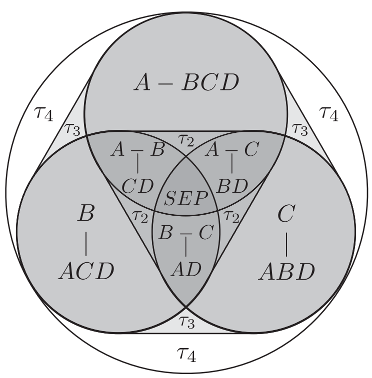


FIG. 1. Simplified, schematic structure of 4-partite states: the grey circles depict quantum states that are separable with respect to three different bipartitions (the biseparation $ABC - D$ and biseparations into pairs of subsystems are not shown). States that belong to two of these sets are at most bipartite entangled, and states that belong to all three sets are separable. Any convex sum of bipartite entangled states (depicted by $\tau_2$) is considered at most bipartite entangled, even though the state might not be separable with respect to any bipartition. Similarly, any convex sum of tripartite entangled states (depicted by $\tau_3$) is considered at most tripartite entangled. Only states that can not be obtained as a convex sum of at most tripartite entangled states are 4-partite entangled.

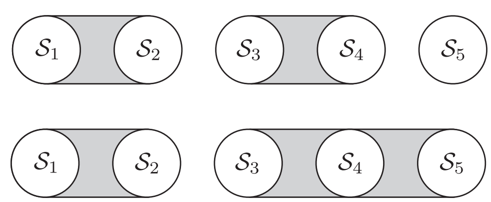


FIG. 2. Schematic representation of two pure five-partite states: a bipartite entangled (top) and a tripartite entangled (bottom). The bipartite entangled state is separable with respect to the 2-bipartitions $\mathcal{S}_1\mathcal{S}_2 - \mathcal{S}_3\mathcal{S}_4\mathcal{S}_5$ and $\mathcal{S}_1\mathcal{S}_2\mathcal{S}_5 - \mathcal{S}_3\mathcal{S}_4$. Indeed, any pure bipartite entangled five-partite state is biseparable with respect to at least two 2-bipartitions. Pure tripartite entangled five-partite states on the other hand can be separable with respect to one 2-bipartition only.

TABLE I. Specific choices for the weight factors $a_i^{(k,n)}$ that define valid criteria $\tau_{k,n}$ to detect $k$-partite entanglement in an $n$-partite system through Eq. (2). Each vector in the table contains the elements $[a_{n/2}^{(k,n)}, \ldots, a_1^{(k,n)}]$ (for even $n$) and $[a_{(n-1)/2}^{(k,n)}, \ldots, a_1^{(k,n)}]$ (for odd $n$). The upper left half ranges from $n = 7$ to $n = 5$, the lower right from $n = 4$ to $n = 2$. Values of $a_i^{(k,n)}$ for $n$ up to 12 can be found in the Supplemental Material [20].

| | $n = 7$ | $n = 6$ | $n = 5$ | |
|---|---|---|---|---|
| $k = 2$ | [0,0,1/35] | [0,0,1/10] | [0,1/10] | |
| $k = 3$ | [0,0,1/3] | [0,1/3,0] | [0,1/2] | |
| $k = 4$ | [0,0,1/2] | [0,1/3,1] | [0,1] | |
| $k = 5$ | [0,0,1] | [0,1,1] | [1,1] | |
| $k = 6$ | [0,1,1] | [1,1,1] | | |
| $k = 7$ | [1,1,1] | | [1,1] | $k = 4$ |
| | | [1] | [0,1] | $k = 3$ |
| | [1] | [1/3] | [0,1/3] | $k = 2$ |
| | $n = 2$ | $n = 3$ | $n = 4$ | |

criterion. Similarly, for $n > 8$ $a_i^{(3,n)} = \delta_{i4}/[m(m-1)/2]$ typically leads to an even stronger criterion. Since picking a good choice for the weight factors helps to identify good criteria, we refrain from providing a systematic description for the construction of the $a_i^{(k,n)}$, but rather depict choices for $n \leq 7$ that we found to yield good results in Table I; similar choices for $n$ up to 12 are available in the Supplemental Material [20]. The functions $\tau_{k,n}$ with these specific coefficients then define a full hierarchy of necessary separability criteria for any system size $n$.

As is the case for any attempt to detect entanglement beyond $2 \times 3$-dimensional systems [6], a tool can either identify entanglement or it can identify separability, but there is none that can assert with certainty whether a state is entangled or separable. Also here a nonpositive value of $\tau_{k,n}$ does not necessarily imply that the considered state was not $k$-partite entangled, but it could also be due to the fact that $\tau_{k,n}$ is not strong enough to identify the targeted entanglement in the specific state. If the latter is the case, one can improve $\tau_{k,n}$ provided there are additional properties that can be exploited. Here we would like to demonstrate this with the example of $W$ states [21], i.e., states with a single excitation $|W\rangle = \sum_i w_i |i\rangle$, where $|i\rangle$ is a shorthand notation for the state with the $i$th subsystem in its excited state and all other subsystems in their ground state. These states attract particular attention since they occur naturally in excitation transport processes [16,22], and they also permitted the observation of genuine multipartite entanglement of an eight ion string [23].

If such a $W$ state is biseparable with respect to an $i$-bipartition, so that $|W\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$, then one of the components $|\phi_{1/2}\rangle$ needs to be completely separable, because otherwise there would be a finite amplitude for two excitations [24]. Consequently, biseparability with respect to an $i$-bipartition ($i \leq n/2$) implies biseparability with respect to at least $i$ one-bipartitions, $i(i-1)/2$ two-bipartitions, and similarly for larger bipartitions. These additional separability properties permit to identify significantly lower values for the weights $a_i^{(k,n)}$ than those for the general states as given in Table I. In contrast to the above, where we found strong criteria based on bipartions of $\simeq n/2$ subsystems, in the case of $W$ states it is rather advantageous to focus on 1-bipartitions: the weights $a_i^{(k,n)} = \delta_{i1}/[n-(k-1)]$ for $k \neq 2$ and $a_i^{(k,n)} = \delta_{i1}/n$ for $k = 2$ provide a strong hierarchy $\tau_{k,n}^w$ for $W$ states. In a similar fashion, the present criteria can also be adjusted for different classes of states, such as more general Dicke states [25] or potentially states with permutation symmetries [26].

So far we have discussed the hierarchies in a rather abstract setting, assuming the existence of functions that satisfy the above list of properties (i) to (iv). Let us become more specific now and present a possible choice of such functions. It is based on the fact that a twofold tensor product $|\Psi\rangle \otimes |\Psi\rangle$ of a state with itself features very specific invariance properties if $|\Psi\rangle$ is not genuinely $n$-partite entangled [27]: $|\Psi\rangle$ is biseparable with respect to a biseparation that divides the system in the components $A$ and $B$ if and only if the twofold state $|\Psi\rangle \otimes |\Psi\rangle$ is invariant under the permutation that permutes the two $A$ components (or, analogously, the $B$ components). Taking $f$ to be a function $g$ of $|\Psi\rangle \otimes |\Psi\rangle$ and $f_{ij} = g(\Pi_{ij}|\Psi\rangle \otimes |\Psi\rangle)$, where $\Pi_{ij}$ is the permutation that permutes the $A$ components associated with the $j$th $i$-bipartition makes sure that condition (iii) is satisfied. Condition (i), i.e., convexity of $\tau_{k,n}$, is in general difficult to achieve, but

$$
\begin{aligned}
f &= \sqrt{\langle \Phi_S | \boldsymbol{\Pi} \varrho \otimes \varrho | \Phi_S \rangle} \quad \text{and} \\
f_{ij} &= \sqrt{\langle \Phi_S | \Pi_{ij} \varrho \otimes \varrho \Pi_{ij}^{\dagger} | \Phi_S \rangle}
\end{aligned}
\tag{3}
$$

with the global permutation $\boldsymbol{\Pi}$ and a product vector $|\Phi_S\rangle$, are convex, respectively, concave [8,9].

For pure states $f$ coincides with $\sqrt{\langle \Phi_S | \varrho \otimes \varrho | \Phi_S \rangle}$ ($|\Psi\rangle \otimes |\Psi\rangle$ is invariant under $\boldsymbol{\Pi}$), so that condition (iii) is satisfied, and the present specific choices for $f$ and $f_{ij}$ are indeed non-negative. As long as $a_i^{(k,n)}$ are non-negative as they should be according to condition (iv), Eqs. (2) and (3) with the weight factors $a_i^{(k,n)}$ such as those given in Table I provide a valid realization of a hierarchy following conditions (i) through (iv).

We have tested the performance of these hierarchies for different, exemplary cases, comparing it with previously known criteria [4,28] for 4- and 6-partite spin-squeezed states and 4-partite $W$ states. This comparison is shown in section A of the Supplemental Material [20]. This test demonstrates how the hierarchies $\tau_{k,n}$ and $\tau_{k,n}^w$ often outperform prior techniques, especially in the presence of strong mixing. This is remarkable, in particular, since existing criteria have been specifically tailored to address entanglement properties of a given class of states, whereas we just varied the coefficients $a_i^{(k,n)}$ retaining the same analytic form. By a systematic application of the hierarchy $\tau_{k,n}$ to a given system of interest it is possible to gain a deep insight in its entanglement structure, as noticeable for the exemplary case of spin-squeezed states in the Supplemental Material [20], where a rich underlying multipartite entanglement landscape is uncovered in a broad interval of the spin-squeezing parameter where previously only bipartite entanglement was detected.

As argued above, the possibility to detect $k$-partite entanglement for varying $k$ also provides a refined insight in entanglement dynamics. This is substantiated in section B of the Supplemental Material [20] with the investigation of a fully connected 12-partite graph state undergoing a dephasing evolution, where it is emphasized how the different types of $k$-partite entanglement decay on different time scales.

The possibility to explore the dynamics of arbitrary $k$-partite entanglement in turn enables to uncover relevant physical features of a given system, as we exemplify with the verification of three-body interactions [29] in section C of the Supplemental Material [20].

These examples underline the usefulness of the present hierarchies; the specific framework of permutation operators that we have used for the explicit construction of the present hierarchy is by no means the sole way to arrive at such a hierarchy. Many other typically employed tools, such as entanglement witnesses [30] or positive maps [31] bear potential for a systematic construction. Also, whereas we have focussed here on the classification of $k$-partite entanglement, a classification in more refined classes according to local operations and classical communication (LOCC)-inequivalence [21,32] seems feasible.

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