Delayed Capillary Breakup of Falling Viscous Jets

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Thin jets of viscous fluid like honey falling from capillary nozzles can attain lengths exceeding 10 m before breaking up into droplets via the Rayleigh-Plateau (surface tension) instability. Using a combination of laboratory experiments and WKB analysis of the growth of shape perturbations on a jet being stretched by gravity, we determine how the jet's intact length l_b depends on the flow rate Q, the viscosity η, and the surface tension coefficient γ. In the asymptotic limit of a high-viscosity jet, $l_b \sim (g Q^2 \eta^4 / \gamma^4)^{1/3}$, where g is the gravitational acceleration. The agreement between theory and experiment is good, except for very long jets.

The breakup of liquid jets due to the action of surface tension is a classic fluid-mechanical instability, first explained theoretically by Plateau [1] and Rayleigh [2]. A familiar example of it is a thin stream of water flowing steadily from a faucet, which breaks up into droplets after a distance ≈ 10 cm. If, however, the water is replaced by a much more viscous fluid like honey, the jet can attain lengths of 10 m or more before breaking. This is paradoxical: Theory [3] predicts that the weight of fluid elements in a long viscous jet is balanced by the vertical momentum flux (inertia) over most of the jet's length and that the viscous force that resists the stretching of the jet is negligible in comparison. But, if this is so, how can the viscosity influence the breakup length? Senchenko and Bohr [4] attempted to answer this question by analyzing the growth of small perturbations of the radius of a viscous jet that is strongly stretched and thinned by gravity. But, their conclusion that the growth rate of the perturbations is independent of viscosity only deepens the paradox instead of resolving it.

In this Letter, we report the first systematic experimental investigation of the breakup length of falling viscous jets and propose a new theory that explains how viscosity acts to delay jet breakup. To set the stage, we recall that fluids falling from circular nozzles typically exhibit three distinct regimes as a function of the flow rate [5]. At very low flow rates, a "periodic dripping" regime occurs in which drops of constant mass detach periodically at a downstream distance comparable to the nozzle diameter. As the flow rate increases, a transition to a "dripping faucet" regime occurs in which the mass of the detaching drops varies quasiperiodically or chaotically. Finally, as the flow rate is increased further, a transition occurs to a "jetting" (J) regime in which a steady jet emerges from the nozzle and breaks up further downstream. Our focus here is on the length of the intact portion of the jet in this regime.

In the literature, the dependence of the breakup length on the flow rate and fluid properties such as the surface tension has been extensively studied for high-speed jets in quiescent or coflowing fluids [6,7]. By contrast, viscous jets falling under gravity have been the subject of only a few experimental [8,9] and theoretical [4,9–11] studies, none of which arrived at a prediction for the breakup length as a function of the flow rate and the fluid properties.

Experiments.—We used silicon oils with densities $\rho = 963$–974 kg m^{-3}, surface tension coefficient $\gamma = 0.021$ N m^{-1}, and viscosities $\nu = 50$–27 800 cS. A thin vertical jet was generated by ejecting the oil downward through a nozzle of diameter $2r_0 = 2$–4 mm at a constant flow rate with a range $Q = 0.0036$–1.4 ml/s, using either a syringe pump controlled by a stepper motor or an open reservoir with an adjustable valve at the bottom. The reservoir was sufficiently large (14 cm × 14 cm wide and 20 cm deep) that the flow rate was constant to within ±2% during all of the experiments. To eliminate the influence of air drag on longer jets (breakup length $l_b > 2.5$ m), we enclosed the nozzle and the jet in a cylindrical vacuum chamber with inner diameter 19 cm and length ≤ 7.5 m. The bottom portion (2 m) of the cylinder was transparent to permit observation. A partial vacuum was created inside the cylinder using a Siemens rotary vacuum motor, allowing even the longest jets ($l_b = 7.5$ m) to remain perfectly straight.

We observed three distinct regimes of behavior of the ejected fluid, including the periodic dripping regime at very low flow rates and the J regime at high flow rates. At intermediate flow rates, however, we did not observe the dripping faucet regime but rather an oscillatory "pulsating" regime. Here, the jet had a reasonably steady shape, especially near the nozzle, and broke up at a well-defined distance that greatly exceeded the nozzle diameter.

However, small periodic oscillations of the jet's shape about the mean diameter occurred, corresponding to the absolute instability identified by Ref. [9].

We measured the breakup length l_b for a total of 87 experiments, including 67 in the jetting regime and 20 in the pulsating regime. In some experiments, we detected the point of breakup by moving a rapid camera step by step along the jet as it thins. In most cases, however, we first located the breakup point approximately by eye with the help of a stroboscope and then used a rapid camera at this location to make a more precise measurement. Figure 2 shows l_b as a function of flow rate for three different viscosities. As one expects intuitively, the breakup length is an increasing function of both the flow rate and the viscosity.

Dimensional analysis.—The first step toward a more quantitative understanding is a dimensional analysis. The breakup length l_b depends on the viscosity ν, the surface tension coefficient γ, the density ρ, the flow rate Q, the gravitational acceleration g, and the nozzle radius r_0. Buckingham's Π theorem [12] then implies $\Pi_b = \text{fct}(\Pi_\eta, \Pi_\gamma, \Pi_r)$, where

$$\Pi_b = l_b\left(\frac{g}{Q^2}\right)^{1/5}, \qquad \Pi_\eta = \frac{\eta}{\gamma}(Qg^2)^{1/5},$$
$$\Pi_\gamma = \frac{\gamma}{\rho}(Q^4 g^3)^{-1/5}, \qquad \Pi_r = r_0\left(\frac{g}{Q^2}\right)^{1/5},$$
$$\tag{1}$$

where $\eta = \rho\nu$.

WKB analysis.—Because the structure of the jet varies slowly in the axial direction, the growth of perturbations can be treated using a WKB-type approach in which disturbances locally have the form of plane waves [11]. The starting point is the equations governing plug flow in a slender vertical jet of viscous fluid [13]:

$$\partial_t A + (Av)' = 0, \tag{2a}$$
$$\rho A(\partial_t v + vv') = 3\eta(Av')' + \rho gA - \gamma A\kappa', \tag{2b}$$
$$\kappa = \frac{1}{r(1 + r'^2)^{1/2}} - \frac{r''}{(1 + r'^2)^{3/2}}, \tag{2c}$$

where $r(z, t)$ is the jet's radius, $A = \pi r^2$, $v(z, t)$ is the axial (vertical) velocity, and κ is the mean curvature of the jet's outer surface. The primes denote differentiation with respect to the distance z beneath the nozzle. Equations (2a) and (2b) express conservation of mass and momentum, respectively. The three terms on the right side of (2b) represent the viscous force that resists stretching, the weight of the fluid, and the surface tension force, respectively, all per unit length of the jet.

In the absence of perturbations, the steady flow of the jet is governed by (2) with $\partial_t = 0$. A general analytical solution of these equations was obtained by Ref. [3] in the limit of no surface tension ($\gamma = 0$). The corresponding axial velocity $v(z)$ is shown in Fig. 1 for two values of the normalized ejection speed $\hat{v}_0 = v_0/(3g\nu)^{1/3}$. A clear
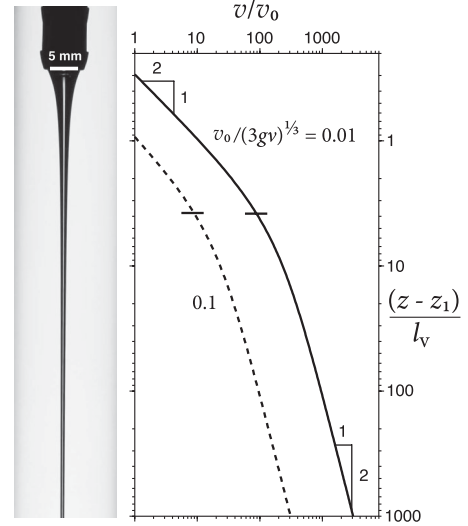

FIG. 1.   Left: A jet of silicon oil with viscosity $\nu = 3500$ cS falling at a volumetric rate $Q = 0.29$ ml s^{-1} from a nozzle with an internal diameter $d = 5$ mm. Right: Axial velocity $v(z)$ in a falling jet without surface tension [3], where $l_\nu = (\nu^2/g)^{1/3}$ and $z = z_1 \equiv -(6\nu v_0/g)^{1/2}$ is a virtual origin above the nozzle. Short horizontal lines indicate the boundary between the viscosity-dominated (above) and inertia-dominated (below) parts of the jet.

distinction is evident between the jet's upper part, where the weight of the fluid is balanced primarily by the viscous force that resists stretching, and its lower part, where the weight is balanced by inertia. The boundary between the two is the point where the viscous and inertial terms in (2b) are equal and occurs at a distance $B(\nu^2/g)^{1/3}$ from the nozzle, where $B = B(\hat{v}_0) \leq 5.0$. Because $l_b \approx 50$–$500(\nu^2/g)^{1/3}$ in our experiments, breakup always occurs in the inertia-dominated part of the jet. The prefactor B drops to zero for $\hat{v}_0 \geq 1.219$, meaning that the weight is then balanced primarily ($> 50\%$) by inertia everywhere in the jet.
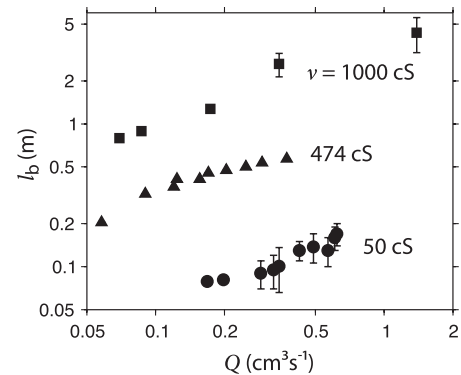

FIG. 2.   Breakup length L in the jetting regime as a function of flow rate Q for three different viscosities and the same nozzle diameter: $2r_0 = 2$ mm. For clarity, only error bars $>15\%$ are shown.

The analytical solution of Ref. [3] can also be used to estimate the magnitude of the neglected surface tension term in (2b) relative to inertia and the viscous force. For the parameter values of our experiments, it turns out that surface tension is negligible in the inertia-dominated part of the jet but not in the viscosity-dominated part. However, because the viscosity-dominated part of the jet is very short compared to the breakup length, we are justified in using the solution of Ref. [3] as our base state.

To model the fluctuating environment surrounding the jet, we introduce small perturbations with different initial wave numbers k_0 at different points along the jet, i.e., at different times t_0, since the fluid element in question exited the nozzle. Each of these perturbations will grow to $O(1)$ amplitude at some distance $z_b(k_0, t_0)$ from the nozzle, at which point the jet will break. We posit that the observed breakup length l_b is the minimum value of the function $z_b(k_0, t_0)$.

Next, we recall that the exponential growth rate σ of small long-wavelength perturbations of wave number k on the surface of a jet of constant radius r is [13]

$$\sigma = \left(\frac{\gamma}{r^3 \rho}\right)^{1/2} \left[\left(\frac{\hat{k}^2(1 - \hat{k}^2)}{2} + \frac{9}{4}\Pi_{\mathrm{Oh}}^2 \hat{k}^4\right)^{1/2} - \frac{3}{2}\Pi_{\mathrm{Oh}}\hat{k}^2\right], \quad (3)$$

where $\hat{k} = kr$ and $\Pi_{\mathrm{Oh}} = \nu\sqrt{\rho/\gamma r}$ is the Ohnesorge number. We now assume that (3) applies at all points along the jet if $r = r(z)$ and $k = k(z)$ are interpreted as the *local* values of the jet radius and perturbation wave number, respectively. Because the base flow stretches fluid elements at a rate $A^{-1}\dot{A}$, $r(z)$ and $k(z)$ are related to their values at the initial position z_0 by $k(z)r^2(z) = k(z_0)r^2(z_0)$. To obtain the total growth of the perturbation, the growth rate (3) must be integrated along Lagrangian paths, taking into account the variation of r and k. The integrated growth rate is [14]

$$\int_{t_0}^{t} \sigma(k_0, \tau)d\tau = \frac{\pi}{Q}\int_{z_0}^{z} r^2(\zeta)\sigma[r(\zeta), k(\zeta)]d\zeta$$
$$\equiv s(k_0, z_0, z), \quad (4)$$

where $k_0 = k(z_0)$ and the time integral has been transformed to a spatial one using $d\tau = d\zeta/v \equiv \pi r^2 d\zeta/Q$. The total growth of perturbations is the exponential of (4).

Now, suppose that breakup occurs when the quantity $s(k_0, z_0, z)$ reaches a critical value s_{cr}. Because $z_b = z_b(k_0, z_0)$, we have the implicit equation $s_{\mathrm{cr}} = s[z_0, k_0, z_b(z_0, k_0)] \equiv \bar{s}(z_0, k_0)$. Differentiating this equation with respect to z_0, we obtain

$$\frac{\partial \bar{s}}{\partial z_0} = \frac{\partial s}{\partial z_0} + \frac{\partial s}{\partial z_b}\frac{\partial z_b}{\partial z_0} = 0. \quad (5)$$

However, the condition of minimal breakup length requires $\partial z_b/\partial z_0 = 0$, and so (5) implies $\partial s/\partial z_0 = 0$. This derivative can now be evaluated using the definition (4) for s, noting that the dependence on z_0 enters only through the lower limit of integration. We thereby find that the optimal growth rate at the initial position z_0 is $\sigma[r(z_0), k_0] = 0$, whence (3) implies that the optimal initial wave number is $k_0 = 1/r(z_0)$. We can therefore write the integrated growth rate as a function of z_0 and z_b alone, viz.,

$$s(z_0, r^{-1}(z_0), z_b) \equiv S(z_0, z_b). \quad (6)$$

Next, differentiate the equation $\bar{s}(z_0, k_0) = s_{\mathrm{cr}}$ with respect to k_0 to obtain

$$\frac{\partial \bar{s}}{\partial k_0} = \frac{\partial s}{\partial k_0} + \frac{\partial s}{\partial z_b}\frac{\partial z_b}{\partial k_0} = 0. \quad (7)$$

However, the condition of minimal breakup length requires $\partial z_b/\partial k_0 = 0$, whence (7) reduces to $\partial s/\partial k_0 = 0$. This in turn implies $\partial S/\partial z_0 = 0$ because $k_0 = 1/r(z_0)$. The problem of finding the most dangerous perturbation therefore reduces to solving the two simultaneous equations

$$S(z_0, z_b) = s_{\mathrm{cr}}, \qquad \frac{\partial S}{\partial z_0}(z_0, z_b) = 0 \quad (8)$$

for z_0 and z_b, which we did using standard MATLAB routines.

To compare the theoretical predictions with our observations, we first plot the dimensionless breakup lengths Π_b as a function of the dimensionless viscosity Π_η for our 87 experiments (Fig. 3). To calculate a theoretical curve to compare with these data, we fix $\Pi_\gamma = 1.0$ and $\Pi_r = 1.5$, the (logarithmic) mean values of those parameters for the experiments. We then adjust s_{cr} iteratively until the theoretical curve best fits the data for $\log_{10}\Pi_\gamma \leq 1.3$. The result is the solid line labeled Π_b in Fig. 3 and corresponds to $s_{\mathrm{cr}} = 8.86$. The fit to the data is good for $\Pi_\gamma \leq 1.3$ (68 experiments) but poor for the 19 experiments with larger values, all but three of which correspond to long jets with $l_b > 3$ m. We speculate that this is due to the extreme thinness (as low as $r = 7$ μm) attained by these long jets, which may render them more susceptible to perturbations than the theory predicts.

When $\Pi_\eta \leq 3$, the curves of Π_b vs Π_η predicted by the theory have a significant dependence on Π_γ, which varies over about 2 orders of magnitude among our experiments (color scale in Fig. 3). To illustrate that dependence, Fig. 3 also shows theoretical curves calculated for $\Pi_\gamma = 0.25$ (dotted line) and 8.0 (dashed line), both with $s_{\mathrm{cr}} = 8.86$. The theoretical curves also depend in principle on Π_r, but that dependence is negligible for the parameters of our experiments.

Finally, Fig. 3 also shows the theoretically predicted distance z_0 at which a perturbation is most dangerous, nondimensionalized as $\Pi_z = z_0(g/Q^2)^{1/5}$. It is smaller than the dimensionless breakup length Π_b by 1–3 orders of magnitude, depending on the value of Π_η. We now ask whether z_0 is within the inertia-dominated or the viscosity-dominated portion of the jet's base state. By comparing the
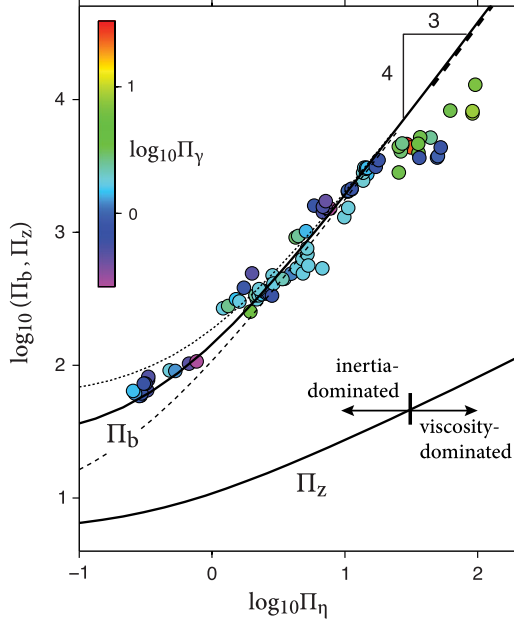
FIG. 3 (color online). Comparison of observed and theoretically predicted breakup lengths. The circles represent observed dimensionless breakup lengths $\Pi_b = l_b(g/Q^2)^{1/5}$ for 87 experiments, with their values of $\Pi_\gamma = (\gamma/\rho)(Q^4 g^3)^{-1/5}$ indicated by colors. The solid line labeled "Π_b" represents the predicted breakup length for $\Pi_\gamma = 1.0$, $\Pi_r = 1.5$, and $s_{cr} = 8.86$. The dotted and dashed lines are the same as the solid line, but for $\Pi_\gamma = 0.25$ and 8.0, respectively. The dashed line at the upper right represents asymptotic expression (9) with $s_{cr} = 8.86$. The solid line labeled "Π_z" represents the predicted dimensionless distance $\Pi_z = z_0(g/Q^2)^{1/5}$ at which the perturbation introduced is most dangerous. The vertical bar shows where z_0 moves from the inertia-dominated to the viscosity-dominated portion of the jet as Π_η increases.

curve Π_z (Π_η) in Fig. 3 with the analytical solution of Ref. [3] for the base state, we find that z_0 moves from the inertia-dominated part of the jet for small Π_η to the viscosity-dominated part for large Π_η. The point of transition is indicated by the short vertical bar in Fig. 3.

High-viscosity limit.—A simple scaling argument yields an asymptotic expression for the breakup length in the high-viscosity limit $\Pi_{Oh} \gg 1$, where the Rayleigh-Plateau growth rate (3) reduces to $\sigma \sim \gamma/(\eta r)$. Now, because most of the jet is in the inertia-dominated regime (free fall), its radius at a distance H below the nozzle is $r \sim (Q^2/gH)^{1/4}$. The growth rate at this distance is therefore $\sigma \sim (\gamma/\eta)(gH/Q^2)^{1/4}$. Now, the time required for a fluid element to fall through the distance H is $\tau \sim (H/g)^{1/2}$. Breakup occurs at the distance $H \equiv l_b$, where the Rayleigh growth time σ^{-1} becomes smaller than the fall time, viz.,

$$l_b = C\left(\frac{gQ^2\eta^4}{\gamma^4}\right)^{1/3} \quad \text{or} \quad \Pi_b = C\Pi_\eta^{4/3}, \qquad (9)$$

where C is a constant. C can be determined from our WKB analysis by expanding the integral expression for S in the limit $\Pi_{Oh} \gg 1$. This permits (8) to be solved analytically for l_b, yielding an expression of the form (9) with $C = (9s_{cr})^{4/3}/(2\pi^{2/3}) \equiv 4.36s_{cr}^{4/3}$. The dashed line in Fig. 3 shows the asymptotic expression (9) with $s_{cr} = 8.86$.

Conclusion.—The resolution of the paradox pointed out in the introduction is now clear: Viscosity plays completely independent roles in the axial momentum balance of the steady basic state and in the growth of perturbations about that state. The analytical solution of Ref. [3] for the basic state shows that viscous forces are negligible in the inertia-dominated part of the jet $z \gg (\nu^2/g)^{1/3}$, which represents $>90\%$ of the jet's length in most of our experiments. However, this does not imply that the effect of viscosity can be neglected in the expression (3) for the Rayleigh-Plateau growth rate. That expression has two limits, depending on the Ohnesorge number $\Pi_{Oh} = \nu\sqrt{\rho/\gamma r}$: a viscosity-independent limit $\sigma \sim (\gamma/\rho r^3)^{1/2}$ for $\Pi_{Oh} \ll 1$ and a (less familiar) viscosity-dominated limit $\sigma \sim \gamma/\eta r$ for $\Pi_{Oh} \gg 1$. To determine which limit is relevant for our experiments, we used the solution of Ref. [3] to calculate $\Pi_{Oh}(r = r_b) \equiv \Pi_{Oh}^b$ for each experiment, where r_b is the jet radius at the distance $z = z_b$ where the jet breaks up. We thereby find that $\Pi_{Oh}^b \in [0.65, 2160]$ and that 66 of our 87 experiments have $\Pi_{Oh}^b > 10$. It is therefore not surprising that the viscosity has a strong influence on the breakup lengths we observe.

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