Fluctuations of $1/f$ Noise and the Low-Frequency Cutoff Paradox

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Recent experiments on blinking quantum dots, weak turbulence in liquid crystals, and nanoelectrodes reveal the fundamental connection between $1/f$ noise and power law intermittency. The nonstationarity of the process implies that the power spectrum is random—a manifestation of weak ergodicity breaking. Here, we obtain the universal distribution of the power spectrum, which can be used to identify intermittency as the source of the noise. We solve in this case an outstanding paradox on the non-integrability of $1/f$ noise and the violation of Parseval's theorem. We explain why there is no physical low-frequency cutoff and therefore why it cannot be found in experiments.

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The power spectrum $S(f)$ of a wide variety of physical systems exhibits enigmatic $1/f$ noise [1,2] at low frequencies

$$S(f) \sim \frac{\text{const}}{f^{\gamma}}, \quad \text{where } 0 < \gamma < 2. \tag{1}$$

Starting with Bernamont [3], in the context of resistance fluctuations of thin films, many models of these widely observed phenomena were put forward. Indeed, $1/f$ noise is practically universal, ranging from voltages and currents in vacuum tubes, diodes, and transistors, to annual amounts of rainfall, to name only a few examples. A closer look at the phenomenon reveals several themes which demand further explanation. The first is that $1/f$ noise is not integrable: $\int_{-\infty}^{\infty} S(f)df = \infty$, due to the low-frequency behavior, when $\gamma \geq 1$. This violates the Parseval theorem from which one may deduce that the spectrum of a random process is integrable (see details below). So, how can we find $1/f$ noise in a laboratory if a mathematical theorem forbids it? One simple explanation would be that the phenomenon has a cutoff at some low frequency, namely, that below $f < f_0$ Eq. (1) is not valid. Experimentalists have therefore carefully searched for this cutoff, increasing the measurement time as far as is reasonable: three weeks for noise in metal-oxide-semiconductor field-effect transistor (MOSFET) [4] and 300 years for weather data [5]. No cutoff frequency is observed even after these long measurement times. This is one of the outstanding features of $1/f$ noise. A second old controversy, related to the first, is the suggestion of Mandelbrot [6] that models of $1/f$ noise for $\gamma \geq 1$ should be related to nonstationarity processes, although the nature of this nonstationarity is still an open question [7,8]. Further, experiments find that at least in some cases the amplitude of the power spectrum varies among identical systems measured at different times, but the shape and in particular the values of the exponent γ are quite consistent [1,2,9]. This means that a $1/f$ spectrum is a non-self-averaging observable, at least in some systems.

While these observations where made long ago, the verdict on them is not yet out. However, recent measurements of blinking quantum dots [10–12], liquid crystals in the electrohydrodynamic convection regime [13], biorecognition [14], and nanoscale electrodes [9] shed new light on the nature of $1/f$ noise. These systems, while very different in their nature, reveal a power law intermittency route to $1/f$ noise. This means that power law waiting times in a microstate of the system are responsible for the observed spectrum. This approach was suggested as a fundamental mechanism for $1/f$ noise in the context of intermittency of chaos and turbulence, with the work of Manneville [15]. Subsequently, it has been found in many intermittent chaotic systems [16–19] and has been used successfully as a model for transport in geological formations [20]. For a quantum dot driven by a continuous wave laser, this mechanism means that the dot switches from a dark state to a bright state where photons are emitted and that sojourn times in both states exhibit a power law statistics that is scale free [21,22]. Waiting times probability density functions (PDFs) in these states follow $\psi(\tau) \sim \tau^{-(1+\alpha)}$ and $0 < \alpha < 1$. The dynamics is scale free because the average sojourn times diverge, and we expect weak ergodicity breaking [23,24]. This means that the power spectrum remains a random variable even in the long time limit [25,26].

Here, we investigate the non-self-averaging power spectrum and show that indeed this observable exhibits large but universal fluctuations while the estimation of γ is rather robust. Our work gives experimentalists a way to verify whether a data set exhibiting $1/f$ noise belongs to the intermittency class, and this we believe will help unravel the origin of an old mystery of statistical physics. We also remove the paradox based on Parseval's identity, showing that as $t \to \infty$ the integrability remains and that there is no cutoff frequency f_0. So, experimentally searching for this "lost" low frequency might be in vain.

Parseval's identity and $1/f$ *noise.*—We consider a measurement of a random signal $I(t)$ in the time interval $(0, t)$, so that its Fourier transform is $\tilde{I}_t(\omega) = \int_0^t I(t')\exp(-i\omega t')dt'$. The power spectrum $S_t(\omega) = [\tilde{I}_t(\omega)\tilde{I}_t^*(\omega)]/t$ is considered in the long measurement time limit. The ensemble average power spectrum is $\langle S_t(\omega)\rangle$. Note that, in an experiment with one realization of the time series, for example, a measurement of the intensity of a single molecule or a quantum dot, the ensemble average is not performed, although in experiments one introduces smoothing methods which reduce the noise level of the reported power spectrum [27]. More importantly, note that the integral over the power spectrum is

$$\begin{aligned} \int_{-\infty}^{\infty} S_t(\omega)d\omega &= \frac{1}{t}\int_{-\infty}^{\infty} d\omega \int_0^t dt_1 \exp(-i\omega t_1)I(t_1) \\ &\quad \times \int_0^t dt_2 \exp(i\omega t_2)I(t_2) \\ &= \frac{2\pi}{t}\int_0^t I^2(t_1)dt_1, \end{aligned} \tag{2}$$

where we used a well known identity of the delta function $\int_{-\infty}^{\infty} d\omega \exp[-i\omega(t_1 - t_2)] = 2\pi\delta(t_2 - t_1)$ and $S_t(\omega) = S_t(-\omega)$ by definition. For any bounded process, be it ergodic or nonergodic, stationary or nonstationary, $I^2(t) \le (I_{\max})^2$, and hence $\int_{-\infty}^{\infty}\langle S_t(\omega)\rangle d\omega \le 2\pi(I_{\max})^2$. So, the integral is finite for a wide class of processes. As a consequence, the nonintegrable $1/f$ noise is strictly prohibited. The classical way out was to *assume* a violation of the $1/f$ behavior in the limit of $f \to 0$.

The ensemble average of Eq. (2), under the additional assumption that the process reaches a stationary state, reads $\lim_{t\to\infty}\int_{-\infty}^{\infty}\langle S_t(\omega)\rangle d\omega = 2\pi\langle I^2\rangle$. If the system is ergodic, i.e., $\overline{I^2} = \int_0^t I^2(t')dt'/t \to \langle I^2\rangle$, we have for a single trajectory $I(t')$

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} S_t(\omega)d\omega = 2\pi\langle I^2\rangle. \tag{3}$$

Thus fluctuations of the total area under the power spectrum are an indication for ergodicity breaking.

Model.—For simplicity, we consider a two-state model, with a state *up* where $I(t) = I_0$ and *down* with $I(t) = -I_0$. The sojourn times in these states are independently identically distributed random variables with PDFs $\psi(\tau)$. Thus, after waiting a random time in any state (called an *epoch*), the particle chooses the next state to be *up* or *down* with equal probability. The waiting time PDFs have long tails $\psi(\tau) \propto \tau^{-(1+\alpha)}$ with $0 < \alpha < 1$; hence, the averages of the *up* and *down* times are infinite. The Laplace $t \to \lambda$ transform of these PDFs is for small λ: $\hat{\psi}(\lambda) \simeq 1 - (\overline{\tau}\lambda)^\alpha$, where $\overline{\tau}$ is a scaling constant. This is a simple stochastic model of a blinking quantum dot, for which typically $\alpha = 1/2$, although $1/2 < \alpha < 1$ was also reported. A different model would be alternating between the two states. These processes are also known as a random telegraph signal [28].

Note that the random process $I(t)$ with N internal states and possible different waiting time distributions can describe annealed trap models used for glass phenomenology [24] or for continuous time random walks describing the motion of single molecules in live cells [29]. Our results for $N > 2$ are structurally similar to those for $N = 2$ but deserve their own discussion, which will be presented in a longer paper. For these models, we have derived detailed expressions for the power spectrum of the process $I(t)$ and its statistical properties.

Statement of the main results.—For $\alpha < 1$, the expectation value of the spectrum is not constant but decreases with measurement time $\langle S_t(\omega)\rangle \simeq t^{\alpha-1}\sigma_\alpha(\omega)$. Expanding the t-independent function $\sigma_\alpha(\omega)$ for small frequencies ω, one finds a typical nonintegrable $1/f$ noise

$$\langle S_t(\omega)\rangle \simeq C\frac{t^{\alpha-1}}{\omega^{2-\alpha}}. \tag{4}$$

In general, the value $S_t(\omega)$ of the spectrum is a fluctuating quantity even in the $t \to \infty$ limit. The statistical behavior of the general class of processes for large t (for pairwise disjoint $\omega_i \neq 0$) is fully described by the convergence in distribution of

$$\left(\frac{S_t(\omega_1)}{\langle S_t(\omega_1)\rangle}, \ldots, \frac{S_t(\omega_n)}{\langle S_t(\omega_n)\rangle}\right) \to Y_\alpha(\xi_1, \ldots, \xi_n), \tag{5}$$

where Y_α is a random variable of normalized Mittag-Leffler distribution with exponent α whose moments are $\langle Y_\alpha^n\rangle = n!\Gamma(1+\alpha)^n/\Gamma(1+n\alpha)$ [30]. The ξ_i are independent exponential random variables with a unit mean. For $\alpha = 1$, the Mittag-Leffler random variable becomes $Y_1 = 1$, so that the powers $S_t(\omega_i)$ of different frequencies become independent exponentially distributed random variables—a result known for several ergodic random processes [31]. In the case of weak ergodicity breaking ($\alpha < 1$), the whole spectrum has a common random prefactor Y_α which shifts the complete observed spectrum.

Many procedures for the estimation of the spectrum from one finite time realization are designed to suppress the statistical fluctuations due to the uncorrelated random variables ξ_i [27,31]. These cannot account for the fluctuations of Y_α, common to all estimators of a given realization. For these procedures, the prefactor affects all estimated values for the spectrum. However, being a common prefactor, it does not affect the shape of the estimated spectrum so that such features as $1/f$ noise can be detected independently of the realization.

Motivation of the results.—We focus here on the two-state model introduced above. Let τ_i be the ith waiting time and $\chi_i = \pm I_0$ the value taken during this waiting time. We denote by $T_j = \sum_{i=1}^{j-1}\tau_i$ the end of the epochs. If $n(t)$ is the number of completed waiting times up to time t, we can approximate by ignoring the waiting time in progress at t

$$\int_0^t d\tau \exp(i\omega\tau) I(\tau) \simeq \sum_{j=1}^{n(t)} d_j(\omega), \quad \text{with}$$
$$d_j(\omega) = i\chi_j \exp(i\omega T_j)\frac{1-\exp(i\omega\tau_j)}{\omega}. \tag{6}$$

With this approximation, one obtains

$$S_t(\omega) \simeq \frac{1}{t}\sum_{k,l=1}^{n(t)} d_k(\omega) d_l(-\omega). \tag{7}$$

Assuming that $n(t)$ is for large t independent of the waiting time of a single step τ_i and using $\langle \chi_i \chi_j\rangle = \delta_{ij} I_0^2$, we get for the ensemble average [32]

$$\langle S_t(\omega)\rangle \simeq \frac{\langle n(t)\rangle}{t}\langle d_1(\omega) d_1(-\omega)\rangle$$
$$\simeq I_0^2 \frac{\langle n(t)\rangle}{t}\frac{2-\hat{\psi}(i\omega)-\hat{\psi}(-i\omega)}{\omega^2}. \tag{8}$$

It has been shown that $n(t) \simeq Y_\alpha t^\alpha/[\Gamma(1+\alpha)\bar{\tau}^\alpha]$ [33,34]. Therefore,

$$\langle S_t(\omega)\rangle \simeq \frac{I_0^2 t^{\alpha-1}}{\Gamma(1+\alpha)\bar{\tau}^\alpha}\frac{2-\hat{\psi}(i\omega)-\hat{\psi}(-i\omega)}{\omega^2}$$
$$\simeq \frac{2I_0^2\cos(\alpha\pi/2)}{\Gamma(1+\alpha)}\frac{t^{\alpha-1}}{|\omega|^{2-\alpha}} \quad \text{as} \quad \omega \to 0. \tag{9}$$

The last line shows the typical $1/f$ noise [26]. It is important that the observation limit $t\to\infty$ is taken before the frequency limit $\omega \to 0$.

We motivate the main result Eq. (5) with the help of a random phase approximation. The random phase approximation assumes that terms of the form $\exp(i\omega T_j)$ are just random phases, and any average over them vanishes. Especially, $\langle d_{j_1}(\nu_1)\cdots d_{j_n}(\nu_n)\rangle = 0$ if $\nu_1 T_{j_1} + \cdots + \nu_n T_{j_n} \neq 0$ for some T_j's (the ν being $\pm\omega$). Looking at the second moment of Eq. (7) to show the convergence in distribution by the method of moments [35] and using (8),

$$\langle S_t^2(\omega)\rangle \simeq \frac{1}{t^2}\left\langle \sum_{k,l,p,q=1}^{n(t)} d_k(\omega)d_l(-\omega)d_p(\omega)d_q(-\omega)\right\rangle$$
$$\simeq \frac{2}{t^2}\langle n(t)^2\rangle\langle d_1(\omega)d_1(-\omega)\rangle^2 \simeq 2\langle Y_\alpha^2\rangle\langle S_t(\omega)\rangle^2, \tag{10}$$

where we ignored terms with $k=l=p=q$ as there are only $\langle n(t)\rangle$ of them. The factor 2 stems from the fact that the sum in the first line of Eq. (10) has contributions for $k=l$, $p=q$ and for $k=q$, $l=p$. In contrast to this, for the term $\langle S_t(\omega_1)S_t(\omega_2)\rangle$ with $\omega_1 \neq \omega_2$, this symmetry factor will not be present. Following the same steps as in Eq. (10) gives

$$\langle S_t(\omega_1)S_t(\omega_2)\rangle \simeq \langle Y_\alpha^2\rangle\langle S_t(\omega_1)\rangle\langle S_t(\omega_2)\rangle. \tag{11}$$

This shows the equality of the second moments of Eq. (5). The equality of the higher moments follows similarly by using combinatorial methods to determine these symmetry factors. We see that the random number of jumps $n(t)$ is responsible for the Mittag-Leffler fluctuations while the random phases generate the exponential noise [32]. The exact proofs for the general model will be published in a longer paper.

Until now, we considered the case that the measurement starts with a renewal. In several circumstances, it happens that the measurement starts at a time ΔT after the process has started. If the measurement time goes to infinity as ΔT stays constant, one can show that this does not change the result (the first waiting time is a residual lifetime [34,36] which can be neglected for large t). The distribution of $n(t)$ in the case that ΔT and the measurement time are of the same order of magnitude has been considered by one of the authors (E. B.) [37]. As the common prefactor is determined by the statistics of $n(t)$, we conjecture that the Mittag-Leffler statistics is replaced by the results of Ref. [37] in this case.

Numerical results.—We simulated the two-state model with $I_0 = 1$ for different lengths of time series and different α. The waiting times were generated by using a uniformly distributed random number $0 < X \leq 1$ and setting $\tau = c_\alpha X^{-1/\alpha}$. The constant c_α was chosen such that $\langle n(1)\rangle \simeq 10\,000$. The ensemble consists of 10 000 realizations of the time series.

In Fig. 1(a), we have plotted different realizations of $S_t(f)$. The stochastic fluctuations inside and between the

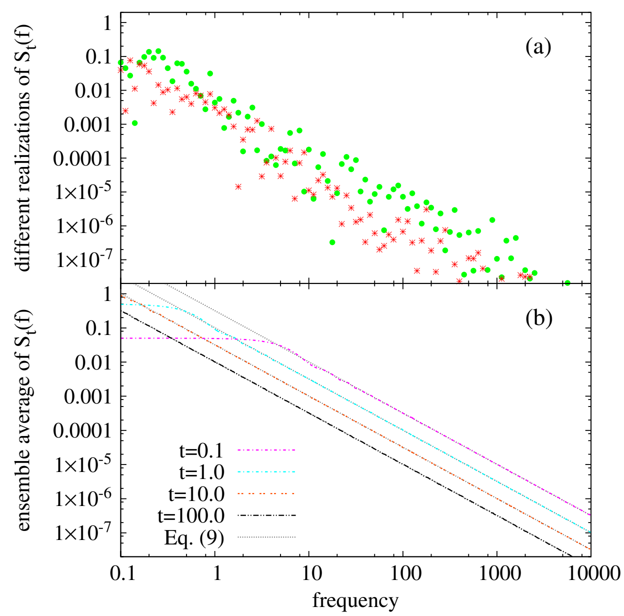

FIG. 1 (color online). (a) $S_t(f)$ plotted for different realizations ($\alpha = 0.5$, $t = 10^2$). Inside each realization one has fluctuations following exponential distributions. Different realizations are shifted with respect to each other due to the random prefactor Y_α. (b) Ensemble average of $S_t(f)$ plotted for different lengths t of the time series. One sees the decay of the spectrum $\langle S_t(f)\rangle \simeq 0.101 t^{-1/2} f^{-3/2}$ [Eq. (9)] both in time and frequency. The crossover frequency is around $f_c \simeq 0.51/t$ [Eq. (14)]. The simulations perfectly match the theory [Eqs. (9) and (13)].

realizations are clearly observable. In Fig. 1(b), the ensemble average of the power spectrum for different lengths is plotted. The $1/f$ spectrum and its decay with observation time is clearly visible. Note that at very low frequencies we find $S_t(\omega) \simeq$ const independent of frequency—an effect we will soon explain.

In a second step, we want to check the statistical properties described by Eq. (5). To isolate the Mittag-Leffler fluctuations, we have calculated the spectrum for a fixed set of N frequencies ω_i and determined the values

$$M = \frac{1}{N}\sum_{i=1}^{N}\frac{S_t(\omega_i)}{\langle S_t(\omega_i)\rangle}. \quad (12)$$

As the exponential distributions are uncorrelated, they average out for sufficiently large N, and the value taken by M should be distributed as the Mittag-Leffler distribution Y_α. We have compared this for different α values. The histogram of M values with the Mittag-Leffler density is shown in Fig. 2 for $\alpha = 0.2$, $\alpha = 0.5$, and $\alpha = 0.8$. A good agreement with the theory is apparent.

These two tests on the numerical data can also be easily applied to measured data by checking for the decay with $t^{1-\alpha}$ and comparing the distribution of the frequency averaged spectrum with a Mittag-Leffler distribution. The existence of these two properties hints at an intermittency caused $1/f$ noise.

Removing the nonintegrability paradox of $1/f$ *noise.*— As mentioned in the introduction, the $1/\omega^{2-\alpha}$ noise is nonintegrable $\int_0^\infty \langle S(\omega)\rangle d\omega = \infty$ due to the low-frequency behavior. This in turn violates the simple bound we have found. To start understanding this behavior, notice that the random phase approximation breaks down when $\omega = 0$, as the phase ωT_n is clearly nonrandom. Hence,

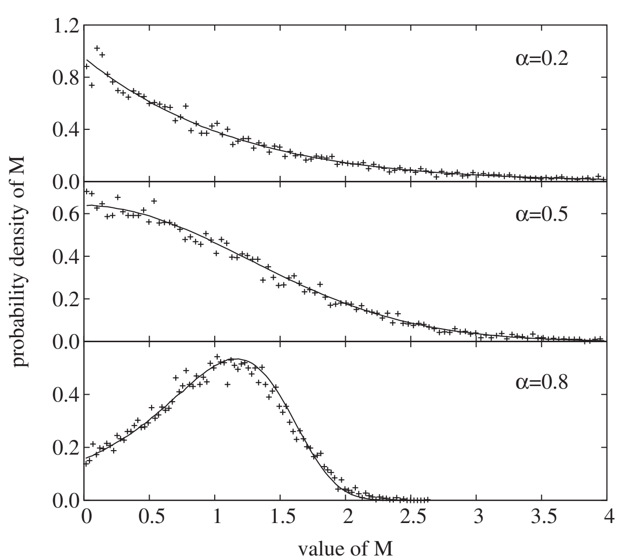

FIG. 2. Distributions of the frequency averaged spectra M [see Eq. (12)]. The lines are the analytic probability densities of the Mittag-Leffler distributions ($t = 10^4$).

the distribution of the power spectrum in Eq. (5) is not valid for $\omega = 0$ and this case must be treated separately.

For $\omega = 0$, we have for a single realization $S_t(0) = \overline{I}^2 t$ with the time average $\overline{I} = \int_0^t I(t')dt'/t$. For ergodic processes, the time average $\overline{I}$ is equal to the ensemble average $\langle I\rangle$. However, for nonergodic processes under investigation, the time average $\overline{I}$ remains a random variable even in the infinite time limit [25,38]. For the two-state process introduced above, we have $\overline{I} = I_0(T^+ - T^-)/t$, where $T^\pm$ is the total time spent in states *up* or *down*. The value of $\overline{I}$ follows an arcsine-like distribution [25,34,38,39]. This simply means that, for a given realization, the system will spend most of the time either in state *up* or in state *down*, and hence $\overline{I}$ is random, which would not be the case for an ergodic process.

This has a consequence for the nonintegrability of the power spectrum. As $S_t(0) = \overline{I}^2 t$, the spectrum at zero frequency tends to infinity, but for any finite measurement time it is finite. For the two-state model, we have on average $\langle S_t(0)\rangle = I_0^2(1-\alpha)t$. So, indeed, *theoretically*, there is a low-frequency cutoff of the divergence of the $1/f$ spectrum, and we now define a crossover frequency ω_c for the transition between the zero-frequency limit, where arcsine statistics takes control (failure of random phase approximation), and higher frequencies, where the Mittag-Leffler statistics takes control. This frequency is defined by merging the two behaviors,

$$\langle S_t(\omega_c)\rangle \simeq C\frac{t^{\alpha-1}}{\omega_c^{2-\alpha}} = \langle S_t(0)\rangle. \quad (13)$$

We see that

$$\omega_c = (C/\langle \overline{I}^2\rangle)^{1/(2-\alpha)}\frac{1}{t}. \quad (14)$$

The values of $\langle \overline{I}^2\rangle$ and C can be obtained from measurements or from theory, for example, for the two-state model $\langle \overline{I}^2\rangle = I_0^2(1-\alpha)$ and $C = 2I_0^2\cos(\alpha\pi/2)/\Gamma(1+\alpha)$. More importantly, we see that the crossover frequency depends on the measurement time as $1/t$. Although at first sight it is surprising, this is the only way how such a crossover can take place in the absence of a characteristic time scale for dynamics: The measurement time itself sets the time scale for crossover. Additionally, $1/t$ appears as the frequency resolution of the discrete Fourier transform typically used in spectral analysis. Importantly, experiments report a lowest frequency at $f = 1/t$.

We see that the increasing measurement time merely stretches the domain of frequency where the $1/f$ noise is observed, which is clearly seen in the numerical simulations [see Fig. 1(b)]. There is no point in increasing measurement time in order to better identify the crossover since a time-independent crossover frequency does not exist. Thus, the $1/f$ noise stretches to the lowest frequencies compatible with measurement time (of the order of $1/t$). This resolves the nonintegrability paradox. The amplitude of the power spectrum itself is also decreasing in time, in such a way that integrability is maintained. Namely,

$$\int_0^\infty \langle S_t(\omega)\rangle d\omega \simeq \langle \bar{I}^2\rangle t\omega_c + \int_{\omega_c}^\infty Ct^{\alpha-1}/\omega^{2-\alpha} d\omega = \frac{2-\alpha}{1-\alpha}(\langle \bar{I}^2\rangle^{1-\alpha} C)^{1/(\alpha-2)} \quad (15)$$

is indeed finite and time independent.

Thus, we conclude that the power spectrum is integrable, as it should be. This seems to indicate the generality of our results since a crossover frequency is only found in few experiments. From a different angle, assuming that the natural frequency is also the limit of measurement $\omega_c \sim 1/t$, we must demand the decrease of the amplitude of power spectrum with time to maintain integrability as required for bounded signals. This, together with the universal fluctuations of $1/f$ noise [Eq. (5)], is a strong fingerprint of power law intermittency. The tools developed here can be tested in a vast number of physical systems.

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