Cold-Atom Quantum Simulator for SU(2) Yang-Mills Lattice Gauge Theory

Erez Zohar,[1] J. Ignacio Cirac,[2] and Benni Reznik[1]

[1]*Raymond and Beverly Sackler Faculty of Exact Sciences, School of Physics and Astronomy, Tel Aviv University, Tel-Aviv 69978, Israel*
[2]*Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, 85748 Garching, Germany*

(Received 14 November 2012; published 21 March 2013)

Non-Abelian gauge theories play an important role in the standard model of particle physics, and unfold a partially unexplored world of exciting physical phenomena. In this Letter, we suggest a realization of a non-Abelian lattice gauge theory—SU(2) Yang-Mills in (1 + 1) dimensions, using ultracold atoms. Remarkably, and in contrast to previous proposals, in our model gauge invariance is a direct consequence of angular momentum conservation and thus is fundamental and robust. Our proposal may serve as well as a starting point for higher-dimensional realizations.

DOI: 10.1103/PhysRevLett.110.125304 PACS numbers: 67.85.Hj, 11.15.Ha

The importance of gauge theories, being at the heart of the standard model of modern particle physics, cannot be overestimated. Gauge fields give rise to the long-range causal interactions between matter particles. However, while QED is an Abelian gauge theory, the strong interactions are described by QCD, a non-Abelian SU(3) Yang-Mills gauge theory. This gives rise to many significant differences. For example, non-Abelian theories involve the effect of quark confinement [1,2], which is responsible for the forces that ''bind quarks together,'' forbidding the existence of free quarks. This gives rise to the familiar structure of hadrons. This effect, as well as other nonperturbative phenomena of (3 + 1)D dimensional non-Abelian models, required the development of new methods and techniques, such as lattice gauge theory [1,3–5]. It has been helpful to study fundamental QCD effects using simpler models that manifest the same essential ingredients. For example, confinement of quarks could be examined within the confining phase of (Abelian) compact QED [1,3,6,7], which already in (2 + 1) dimensions gives rise to flux loops and plaquette interactions. Another approach is to use the (1 + 1)D version of SU(N_c) Yang-Mills theories [such as QCD_2, the (1 + 1)D version of QCD]; (1 + 1)-dimensional models have been used in numerous nonperturbative methods to study the hadronic spectrum of such gauge theories [8–14].

The field of quantum simulations [15–17] has been extensively developed theoretically and experimentally with the aim of advancing new quantum computational approaches to many-body systems, in particular, in the context of condensed matter physics. Recently, it has been realized that quantum simulations could be used also for exploring high energy physics models and effects [18]. Simulations of high energy physics models are in general more demanding, as compared with condensed matter ones for several reasons. Matter and gauge fields are described by fermionic and bosonic fields, and thus the use of several atomic species is needed. Furthermore, the continuum limit of the models must include relativistic symmetries, which can be simulated with nonrelativistic atoms by using lattices, as in lattice gauge theories. There have been several suggestions for the simulation of relativistic field theories involving bosons [19–21], fermions, free or interacting with external (classical) gauge fields [22–24], or fermions interacting with (bosonic quantum) gauge fields in (1 + 1)D [25].

The simulation of *dynamic* gauge theories is even more challenging. First, gauge invariance must be respected. Moreover, it requires a special form of the many-body interactions, which are usually not available in most systems. In particular, lattice models involve plaquette interactions among the links of the lattice. Second, Gauss law (involving gauge bosons and fermions) has to be implemented as a constraint. The special interactions could be obtained as a low-energy effective gauge invariant theory of the original system.

Very recently, realizations of Abelian dynamic gauge theories, employing trapped atoms in optical lattices, have been proposed, simulating (2 + 1)D Kogut-Susskind Abelian compact QED, that manifests confinement of charges, using either Bose-Einstein condensates [26] or single atoms (a truncated version) [27]. These can also be extended to include dynamic matter, as proposed for the (1 + 1)-dimensional Schwinger model [28], which can be compared to exact available solutions, and for a (2 + 1)-dimensional truncated Kogut-Susskind model [29]. Simulations of other Abelian gauge theories [30–32] have been proposed as well.

In this Letter we present a simulation scheme for an SU(2) Yang-Mills theory, where both the fermions (matter) and gauge bosons are dynamical. The main idea is to introduce additional fermionic and bosonic fields (ancillas) to obtain the desired non-Abelian aspect. Remarkably, and in contrast to previous proposals, in our model gauge invariance is fundamental and a direct consequence of angular momentum conservation, making this important property fundamental and robust.

Lattice Yang-Mills theory.—The system we wish to simulate includes a non-Abelian gauge field and dynamic fermionic matter. In ordinary lattice theory the gauge field

degrees of freedom are defined on the lattice's links, whereas the matter fields are located on the vertices [3–5]. The gauge field is represented by unitary matrices U_n^r, whose elements $(U_n^r)_{kl}$ are constructed out of operators in the local (gauge field) Hilbert space. The index n labels the link (according to the vertex from which it emanates), and r the representation. In our SU(2) case, the fundamental representation is $r = 1/2$ and we shall suppress the index r in this case. Thus, $U_n \equiv U_n^{1/2}$ are 2×2 matrices of operators.

Non-Abelian fields generally carry color charges, and hence, unlike in the Abelian case, each link carries *two* different electric color fields, the left and right field characterized by the operators $\{L_{n,a}\}$, $\{R_{n,a}\}$. Their difference along a link can be interpreted as the color charge carried by it. These are, in fact, the left and right generators of the group, and hence they must satisfy the (matrix) algebra of SU(2) [33],

$$[L_a, U^r] = \lambda_a^r U^r; \qquad [R_a, U^r] = U^r \lambda_a^r, \tag{1}$$

where $\{\lambda_a^r\}$, the representation matrices, and $\lambda_a^{r=1/2} = 1/2\sigma_a$. As generators of SU(2), the left and right electric fields satisfy the algebra

$$[L_{n,a}, L_{n,b}] = -i\epsilon_{abc} L_{n,c}; \quad [R_{n,a}, R_{n,b}] = i\epsilon_{abc} R_{n,c}. \tag{2}$$

The left and right generators can be shown to commute with each other, and thus give rise to the same Casimir operator, $\mathbf{L}^2 = \sum_a L_a L_a = \mathbf{R}^2$. Thus, in SU(2) the gauge field Hilbert space on a single link is characterized by three different integer quantum numbers, j, m, m', satisfying

$$\begin{aligned} &\mathbf{L}^2 |jmm'\rangle = \mathbf{R}^2 |jmm'\rangle = j(j+1)|jmm'\rangle, \\ &L_z |jmm'\rangle = m|jmm'\rangle; \qquad R_z |jmm'\rangle = m'|jmm'\rangle. \end{aligned} \tag{3}$$

Finally, the $r = 1/2$ matter fields are introduced as two-component spinors ψ_n defined at the vertices. The local color charges are defined by $Q_{n,a} = 1/2 \sum_{k,l} \psi_{n,k}^\dagger (\sigma_a)_{kl} \psi_{n,l}$. The local gauge invariance is manifested by the conservation of Gauss's law at each vertex, $L_{n,a} - R_{n-1,a} = Q_{n,a}$, for each group index a separately. In local gauge transformations, one picks a group element V_n for each vertex and acts with it on the gauge and matter fields: $\psi_n \to V_n \psi_n$; $U_n \to V_n U_n V_{n+1}^\dagger$. This is a transformation in "group space," i.e., on group indices, and thus, in a gauge invariant Hamiltonian, all the group space indices (of matter and gauge fields) must be fully contracted (effectively "traced out"). Thus, the simplest "pure gauge" terms are of the form $\propto \mathrm{Tr}_{\mathrm{group}}(U_1 U_2 U_3^\dagger U_4^\dagger)$, where the product is of group elements around a plaquette. Such terms give rise to the propagating effects in $(d+1)$ dimensions where $d > 1$, but are absent in $(1+1)$D, where the only possible interactions are gauge-matter ones. In the following, we shall use the staggered fermions method in $(1+1)$ dimensions [34–36], where the gauge invariant Hamiltonian is

$$H = \sum_n \Big[\frac{g^2}{2} \mathbf{L}_n^2 + m(-1)^n \psi_n^\dagger \psi_n + i\beta(\psi_n^\dagger U_n \psi_{n+1} - \mathrm{H.c.}) \Big], \tag{4}$$

where g is the theory's coupling constant and m is the fermion's mass. We shall denote by $|\mathrm{vac}\rangle$ the zeroth-order ground state in the strong coupling limit ($g^2 \gg \beta$, where the β part is treated as a perturbation). This state satisfies $|\mathrm{vac}\rangle = \otimes_{\mathrm{links}} |000\rangle \otimes_{\mathrm{vertices}\ n} |\psi_n^\dagger \psi_n = 1 - (-1)^n\rangle$. The double filling in the odd vertices ("negative mass" vertices) is the "Dirac sea"; in the continuum limit of staggered fermions, two neighboring two-component spinors become a single four-component one.

In order to simulate this Hamiltonian, one has to find an appropriate realization for the group elements $\{U_n\}$ and generators $\{L_{n,a}\}$, $\{R_{n,a}\}$ fulfilling the unitarity and algebra demands [Eqs. (1)–(3)]. The well-known Jordan-Schwinger map [37,38], connecting harmonic oscillators (bosons) and angular momentum can be generalized to mapping between SU(N) and bosonic systems [39,40]. This allows one to express the gauge field operators using bosonic atoms in the prepotential representation [41,42]. (One could also use the quantum link model [43,44], or the generalization with fermionic building blocks [45,46], which provides a finite dimensional Hilbert space approach [47].)

In the prepotential representation, for SU(2), one defines four bosonic species on each link n: the first two are identified by the operators a_1, a_2 on the left side, and the other two b_1, b_2 on the right side, constrained by $N_L = N_R$, where $N_L \equiv a_1^\dagger a_1 + a_2^\dagger a_2$ and $N_R \equiv b_1^\dagger b_1 + b_2^\dagger b_2$. Then, one can express the unitary operators on each link as $U = U_L U_R$, where

$$\begin{aligned} U_L &= \frac{1}{\sqrt{N_L + 1}} \begin{pmatrix} a_1^\dagger & -a_2 \\ a_2^\dagger & a_1 \end{pmatrix}; \\ U_R &= \begin{pmatrix} b_1^\dagger & b_2^\dagger \\ -b_2 & b_1 \end{pmatrix} \frac{1}{\sqrt{N_R + 1}}. \end{aligned} \tag{5}$$

Then, by identifying $j = N_L/2 = N_R/2$, the generators

$$L_a = \frac{1}{2} \sum_{k,l} a_k^\dagger (\sigma_a)_{lk} a_l, \qquad R_a = \frac{1}{2} \sum_{k,l} b_k^\dagger (\sigma_a)_{kl} b_l \tag{6}$$

and $\mathbf{L}^2 = N_L/2(N_L/2+1)$, $\mathbf{R}^2 = N_R/2(N_R/2+1)$, Eqs. (1)–(3) follow.

Pure gauge simulation.—First, we shall discuss the simulation of the gauge field, disregarding the fermions. The simulating system required for that is a set of optical lattices [15]. Each minimum can contain two different (bosonic) atomic species out of four, $A_{1,2}$ or $B_{1,2}$. The A, B minima alternate [see Figs. 1(a) and 1(b)]. We assume that the energy levels of the bosonic modes on each minima are fairly separated, such that we can consider only the lowest one. Since nearest-neighbor minima cannot contain

similar atomic species, tunneling is eliminated, and thus the only remaining interactions are within the minima—scattering and number terms. Tuning the optical parameters and the chemical potential properly, one gets the Hamiltonian

$$H_E = \frac{1}{2}\sum_n \left[g_L \frac{N_{L,n}}{2}\left(\frac{N_{L,n}}{2}+1\right) + g_R \frac{N_{R,n}}{2}\left(\frac{N_{R,n}}{2}+1\right)\right], \tag{7}$$

with $g_R + g_L = g^2$, but since $[H_E, N_{L,n} - N_{R,n}] = 0$, if the constraint $N_{L,n} = N_{R,n}$ is initially fulfilled at all links, we get the desired Hamiltonian, $H_E = g^2/2\sum_n \mathbf{L}_n^2$. This corresponds to a $(1+1)$-dimensional SU(2) pure gauge theory. However, since this system has no dynamics at all, we would like to introduce some dynamic color charges (fermions).

Dynamic fermions simulation.—First, we show how to realize the fermionic mass term. In order to do that, we introduce two-component spinors ψ_n at every vertex [see Fig. 1(c)], i.e., to the left of the A bosonic minima, with an alternating chemical potential, yielding the requested Hamiltonian $H_m = m\sum_n (-1)^n \psi_n^\dagger \psi_n$. This requires the use of a superlattice, as indicated in Fig. 1(c). Next, we introduce the nontrivial interaction term. In order to do that, we introduce ancillary fermionic species, "sitting" in the middle of the links, to the right of the A bosonic minima [see Fig. 1(d)]. On each such "virtual vertex" we define a two-component spinor χ_n with the local Hamiltonian $H_\chi = \lambda\sum_n \chi_n^\dagger \chi_n$. In addition to these spinors, we introduce more bosonic species, $C_{1,2}$ and $D_{1,2}$, whose minima overlap with the ones of $A_{1,2}$ and $B_{1,2}$, respectively [see Figs. 1(a) and 1(b)], serving as reference baths, all prepared in an identical coherent (i.e., Bose-Einstein condensate) state $|\alpha\rangle$, where $\alpha \in \mathbb{R}$, $\alpha \gg 1$.

The required boson-fermion interaction is obtained as a boson-assisted tunneling, where fermionic tunneling is accompanied by an internal boson change, specifically,

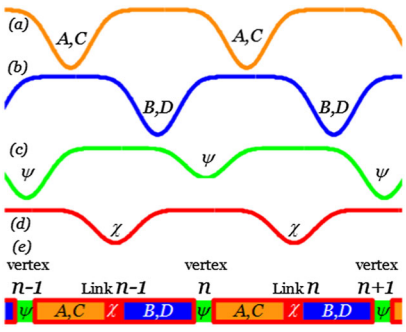

FIG. 1 (color online). A schematic representation of the required bosons (a), (b) and fermions (c), (d) superlattice structure; panel (e) shows schematically the combination of all the species to form the links-vertices structure. We have exaggerated the distance between the wells in order to make the structure more clear.

$$H_f = \frac{\epsilon}{2^{1/4}\alpha}\sum_{n,i,j}[(\psi_n^\dagger)_i(\tilde{W}_{L,n})_{ij}(\chi_n)_j + (\chi_n^\dagger)_i(\tilde{W}_{R,n})_{ij}(\psi_{n+1})_j + \text{H.c.}], \tag{8}$$

where

$$\tilde{W}_L = \begin{pmatrix} a_1^\dagger c_1 & -a_2 c_2^\dagger \\ a_2^\dagger c_2 & a_1 c_1^\dagger \end{pmatrix}; \qquad \tilde{W}_R = \begin{pmatrix} b_1^\dagger d_1 & b_2^\dagger d_2 \\ -b_2 d_2^\dagger & b_1 d_1^\dagger \end{pmatrix}. \tag{9}$$

The overlapping of the Wannier functions of the different atomic species (see Fig. 1) naturally gives rise to such interactions. The minima of the same species are always separated by minima of other ones, and hence tunneling between neighboring same-species minima are unlikely. Thus, only the scattering processes contribute. Their coefficients (such as g^2, the ϵ/α ratio, or γ, which will be defined later) can be tuned by optical Feshbach resonances [48–50].

In order to eliminate undesired processes and keep only the needed ones, one has to carefully select the hyperfine levels of each of the atomic species such that only the desired processes would conserve angular momentum. Thus, gauge invariance will be a fundamental property of the system (and not an effective one), arising from the natural atomic angular momentum conservation. There are many possible choices for the magnetic levels to be used. If one, for example, chooses ^{7}Li atoms for the bosons and ^{40}K atoms for the fermions, there are many choices of m_F values that should work (the selection rules we consider depend only on the magnetic level m_F). A possible choice is $m(A_1) = 3$, $m(A_2) = 2$, $m(B_1) = -3$, $m(B_2) = -2$, $m(C_1) = 1$, $m(C_2) = -3$, $m(D_1) = -1$, $m(D_2) = 3$, $m(\psi_1) = 3/2$, $m(\psi_2) = -3/2$, $m(\chi_1) = 7/2$, $m(\chi_2) = -7/2$ [51]. A detailed discussion can be found in the Supplemental Material [52].

Another term which conserves angular momentum and arises in the proposed setup is $\tilde{H}_f = \gamma\sum_n \psi_n^\dagger \psi_n (N_{R,n-1} + N_{L,n})$, but this can be eliminated by a proper choice of the Hamiltonian parameters, as is explained later. Scattering terms of the form $\chi_n^\dagger \chi_n (N_{L,n} + N_{R,n})$ are also angular momentum conserving and thus possible. However, they vanish in the relevant subspace $\mathcal{M}$, where the dynamics takes place (see below). Finally, the only angular momentum allowed ψ-χ and χ-χ scattering processes vanish in $\mathcal{M}$. The local ψ-ψ scattering should be made as small as possible. Bosonic "undesired" angular momentum allowed processes (C_i-A_j and B_i-D_j scattering) may be eliminated by the use of local linear (number) counterterms (for example, $c_1^\dagger c_1 a_1^\dagger a_1$ may be eliminated using a counterterm $\propto \alpha^2 a_1^\dagger a_1$).

Since $c_i|\alpha\rangle = \alpha|\alpha\rangle$, and $c_i^\dagger|\alpha\rangle \approx \alpha|\alpha\rangle$ (because $\alpha \in \mathbb{R}$, $\alpha \gg 1$, and the same applies for d_i), we get that effectively, within our subspace of interest, we can replace $\tilde{W}_L$, $\tilde{W}_R$ with αW_L, αW_R, where $W_L = \sqrt{N_L + 1}U_L$ and $W_R = U_R\sqrt{N_R + 1}$. Thus, one effectively gets

$$H_f = \frac{\epsilon}{2^{1/4}} \sum_{n,i,j} \Big[\sqrt{N_{L,n}+1}\,(\psi_n^\dagger)_i (U_{L,n})_{ij} (\chi_n)_j + (\chi_n^\dagger)_i (U_{R,n})_{ij} (\psi_{n+1})_j \sqrt{N_{R,n}+1} + \text{H.c.} \Big]. \quad (10)$$

Note that this Hamiltonian is gauge invariant, if we take into account the χ's as well when doing a gauge transformation.

Next, assume that H_χ is the largest energy scale, i.e., $\lambda \gg g^2, m, \epsilon$. In that case, we can treat $H_E + H_m + H_f + \tilde{H}_f$ as perturbations to the large constraint H_χ. If we initially prepare the system with no χ atoms at all, we adiabatically eliminate them and obtain an effective Hamiltonian expansion in this subspace [53], which we denote by $\mathcal{M}$. In first order, we get $H_E + H_m + \tilde{H}_f$. In second order, there are several possible contributions. Each virtual action of H_f requires a second operation of it on the same link, in order to return to $\mathcal{M}$. The first type of processes yields the Hamiltonian $H_f' = -\epsilon^2/\sqrt{2}\lambda\gamma\tilde{H}_f$ (see the Supplemental Material [52] for details). Choosing $\epsilon^2/\sqrt{2}\lambda = \gamma$, we eliminate both $\tilde{H}_f$, H_f' as anticipated above.

The other type of processes are of utmost interest. There, a fermion hops to the middle of a link, and then makes its way to the other side. Mathematically (details in the Supplemental Material [52]), one gets $H_\beta = \frac{\beta}{\sqrt{2}} \sum_n (\psi_n^\dagger \sqrt{N_{L,n}+1}\, U_n \sqrt{N_{R,n}+1}\, \psi_{n+1} + \text{H.c.})$, where $\beta \equiv -\epsilon^2/\lambda$. This yields the full effective Hamiltonian, to second order, $H_{\text{eff}} = H_E + H_m + H_\beta$. The gauge-fermions coupling is folded within H_β. Now, let us redefine the fermionic degrees of freedom, by performing on all the "real" fermions the canonical transformation $\psi_n \to i^n \psi_n$. This transformation leaves H_m invariant, but introduces the missing complex factors to H_β.

Define the strong limit vacuum $|\text{vac}\rangle$, in which the C, D atoms are in the $|\alpha\rangle$ state previously defined, and there are no other bosons all over the lattice. The fermions should be prepared as explained before. All the gauge invariant states can be constructed by operating on $|\text{vac}\rangle$ with products of H_β elements.

Consider the case of small β, comparing to the other energy scales. Then, if one initially starts with the state $|\text{vac}\rangle$, the dynamics will be correct to fifth order in β [52]. A single action of H_β on $|\text{vac}\rangle$ raises the flux on the relevant link to $j = \frac{1}{2}(N_L = N_R = 1)$. Since a link in its ground state contains no A, B bosons, only the creation operators part of U, U_+ would contribute, and one would get, before its operation, $\sqrt{N+1}|\text{vac}\rangle = |\text{vac}\rangle$, and after its action, $\sqrt{N+1}U|\text{vac}\rangle = \sqrt{N+1}U_+|\text{vac}\rangle = \sqrt{2}U_+|\text{vac}\rangle = \sqrt{2}U|\text{vac}\rangle$. Thus, we conclude that $H_\beta|\text{vac}\rangle = i\beta\sum_n(\psi_n^\dagger U_n \psi_{n+1} - \text{H.c.})|\text{vac}\rangle$ as desired. One can also show (as done in the Supplemental Material [52]) that a double operation of H_β on $|\text{vac}\rangle$ is equivalent to the simulated system's case as well, since the flux never exceeds $j = 1/2$ in the second order. In fact, given the initial state $|\text{vac}\rangle$, the flux never exceeds $j = 1/2$ until the sixth order [52], and thus, for small β, one can effectively drop the square roots of number operators and get, for the initial $|\text{vac}\rangle$, a fifth-order accurate simulation of the matter-gauge interactions,

$$H_\beta = i\beta \sum_n (\psi_n^\dagger U_n \psi_{n+1} - \text{H.c.}). \quad (11)$$

Initial state preparation and possible measurements.—Initially, one can prepare the system in the ground state of the strong limit, $|\text{vac}\rangle$. The fermion interactions should be switched off, i.e., $\beta = 0$ (or $\epsilon = 0$, in terms of the fundamental Hamiltonian), for example, by deepening the optical lattice minima. If one turns on the interactions, accurate dynamics will be obtained up to fifth order in β. Alternatively, before turning the interactions on, one can create, using single addressing lasers [54,55], charges with the appropriate flux tubes connecting them. One could either create mesons, whose length should be odd (since we are using staggered fermions and "quarks" and "antiquarks" are on alternating vertices) or baryons, sitting on the same vertex, as described in Ref. [36]. In the case of such initial states, the dynamics will be slightly different than in the simulated model but will manifest, qualitatively, the same effects. After turning the fermionic dynamics on, one can change the parameters m, g, and if it is done adiabatically, one can see the consequences, as long as $\beta < g^2$ (no phase transition is expected in 1 spatial dimension, of course). Measurements can be done by locally probing the number of "real" fermions and A, B bosons over the lattice. Another possibility is to realize the Wilson-loop area law measurement, proposed in Refs. [29,56] for a non-Abelian system.

Before concluding, we shall emphasize again that in this model gauge invariance is fundamental and exact, inherited from angular momentum conservation. This makes the model robust against errors and corrections: as the symmetry is already manifested in the basic Hamiltonian (8), the system cannot leave the gauge invariant subspace.

Generalization to more dimensions.—As mentioned before, lattice gauge theories in $(d+1)$ dimensions $(d > 1)$ contain "magnetic" plaquette terms, unlike the $(1+1)$-dimensional case presented here. Thus, the generalization to higher dimensions requires more complicated techniques [57]. Another important consequence of these terms is that for $d > 1$, the $\sqrt{N+1}$ operators encountered here cannot be avoided, since these closed flux loops raise the flux to $j > 1/2$ in lower orders, making the dynamics less accurate. In this work we have considered the simpler $(1+1)$-dimensional model, although some of the ideas introduced here may be used as a basis to build higher-dimensional versions.

Finally, let us emphasize that it is fair to recognize that the simulation proposed here requires setups that are much more complex than the ones required for the simulation of condensed matter models, something which will make the realization of the present proposal extremely challenging. However, the purpose of our work is to show that this, in principle, is possible, and to trigger the development of the

required experimental techniques, since simulation of gauge theories may have a strong impact well beyond atomic and condensed matter physics.

B. R. acknowledges the support of the Israel Science Foundation, the German-Israeli Foundation, and the European Commission (PICC). J. I. C. is partially supported by the EU project AQUTE. E. Z. acknowledges the support of the Adams Fellowship of the Israel Academy of Sciences and Humanities.

Note added.—Recently, two other proposals for quantum simulations of non-Abelian gauge theories with cold atoms have been suggested, for a strong-coupling rishon-link model [58] and an SU(2) gauge magnet [59].

[1] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[2] A. M. Polyakov, Nucl. Phys. **B120**, 429 (1977).
[3] J. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975).
[4] J. B. Kogut, Rev. Mod. Phys. **51**, 659 (1979).
[5] J. B. Kogut, Rev. Mod. Phys. **55**, 775 (1983).
[6] T. Banks, R. Myerson, and J. Kogut, Nucl. Phys. **B129**, 493 (1977).
[7] S. D. Drell, H. R. Quinn, B. Svetitsky, and M. Weinstein, Phys. Rev. D **19**, 619 (1979).
[8] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).
[9] C. G. Callan, N. Coote, and D. J. Gross, Phys. Rev. D **13**, 1649 (1976).
[10] E. Witten, Commun. Math. Phys. **92**, 455 (1984).
[11] Y. Frishman and J. Sonnenschein, Phys. Rep. **223**, 309 (1993).
[12] K. Hornbostel, S. J. Brodsky, and H.-C. Pauli, Phys. Rev. D **41**, 3814 (1990).
[13] D. J. Gross, I. R. Klebanov, A. V. Matytsin, and A. V. Smilga, Nucl. Phys. **B461**, 109 (1996).
[14] A. Armoni, Y. Frishman, and J. Sonnenschein, Nucl. Phys. **B596**, 459 (2001).
[15] M. Lewenstein, A. Sanpera, and V. Ahufinger, *Ultracold Atoms in Optical Lattices: Simulating Quantum Many-Body Systems* (Oxford University Press, New York, 2012).
[16] I. Bloch, J. Dalibard, and S. Nascimbene, Nat. Phys. **8**, 267 (2012).
[17] R. Blatt and C. F. Roos, Nat. Phys. **8**, 277 (2012).
[18] J. I. Cirac and P. Zoller, Nat. Phys. **8**, 264 (2012).
[19] A. Retzker, J. I. Cirac, and B. Reznik, Phys. Rev. Lett. **94**, 050504 (2005).
[20] B. Horstmann, B. Reznik, S. Fagnocchi, and J. I. Cirac, Phys. Rev. Lett. **104**, 250403 (2010).
[21] S. P. Jordan, K. S. M. Lee, and J. Preskill, Science **336**, 1130 (2012).
[22] A. Bermudez, L. Mazza, M. Rizzi, N. Goldman, M. Lewenstein, and M. A. Martin-Delgado, Phys. Rev. Lett. **105**, 190404 (2010).
[23] O. Boada, A. Celi, J. I. Latorre, and M. Lewenstein, New J. Phys. **13**, 035002 (2011).
[24] L. Mazza, A. Bermudez, N. Goldman, M. Rizzi, M. A. Martin-Delgado, and M. Lewenstein, New J. Phys. **14**, 015007 (2012).
[25] J. I. Cirac, P. Maraner, and J. K. Pachos, Phys. Rev. Lett. **105**, 190403 (2010).
[26] E. Zohar and B. Reznik, Phys. Rev. Lett. **107**, 275301 (2011).
[27] E. Zohar, J. I. Cirac, and B. Reznik, Phys. Rev. Lett. **109**, 125302 (2012).
[28] D. Banerjee, M. Dalmonte, M. Müller, E. Rico, P. Stebler, U.-J. Wiese, and P. Zoller, Phys. Rev. Lett. **109**, 175302 (2012).
[29] E. Zohar, J. I. Cirac, and B. Reznik, Phys. Rev. Lett. **110**, 055302 (2013).
[30] E. Kapit and E. Mueller, Phys. Rev. A **83**, 033625 (2011).
[31] G. Szirmai, E. Szirmai, A. Zamora, and M. Lewenstein, Phys. Rev. A **84**, 011611 (2011).
[32] L. Tagliacozzo, A. Celi, A. Zamora, and M. Lewenstein, arXiv:1205.0496v1.
[33] Or more explicitly, $[L_a, (U^r)_{kl}] = (\lambda^r_a U^r)_{kl}$.
[34] T. Banks, L. Susskind, and J. Kogut, Phys. Rev. D **13**, 1043 (1976).
[35] L. Susskind, Phys. Rev. D **16**, 3031 (1977).
[36] C. Hamer, Nucl. Phys. **B121**, 159 (1977).
[37] P. Jordan, Z. Phys. **94**, 531 (1935).
[38] J. Schwinger, U. S. Atomic Energy Commission Report No. NYO-3071, 1952.
[39] M. Mathur and D. Sen, J. Math. Phys. (N.Y.) **42**, 4181 (2001).
[40] M. Mathur and H. Mani, J. Math. Phys. (N.Y.) **43**, 5351 (2002).
[41] M. Mathur, J. Phys. A **38**, 10 015 (2005).
[42] R. Anishetty, M. Mathur, and I. Raychowdhury, J. Phys. A **43**, 035403 (2010).
[43] D. Horn, Phys. Lett. B **100**, 149 (1981).
[44] P. Orland and D. Rohrlich, Nucl. Phys. **B338**, 647 (1990).
[45] S. Chandrasekharan and U.-J. Wiese, Nucl. Phys. **B492**, 455 (1997).
[46] R. Brower, S. Chandrasekharan, and U.-J. Wiese, Phys. Rev. D **60**, 094502 (1999).
[47] In this representation, one utilizes representations of SU($2N$), which result with a U(N) symmetry, and thus more terms ($\propto \det U$) are required in the Hamiltonian, in order to reduce the symmetry to SU(N).
[48] P. O. Fedichev, Y. Kagan, G. V. Shlyapnikov, and J. T. M. Walraven, Phys. Rev. Lett. **77**, 2913 (1996).
[49] J. L. Bohn and P. S. Julienne, Phys. Rev. A **56**, 1486 (1997).
[50] F. K. Fatemi, K. M. Jones, and P. D. Lett, Phys. Rev. Lett. **85**, 4462 (2000).
[51] State-dependent potentials for different hyperfine levels require laser detunings relatively close to resonance, which may lead to spontaneous emission over longer time scales. In order to avoid spontaneous emission one may use other atomic species.
[52] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.110.125304 for more technical details.
[53] C. E. Soliverez, J. Phys. C **2**, 2161 (1969).
[54] W. S. Bakr, J. I. Gillen, A. Peng, S. Folling, and M. Greiner, Nature (London) **462**, 74 (2009).
[55] C. Weitenberg, M. Endres, J. F. Sherson, M. Cheneau, P. Schausz, T. Fukuhara, I. Bloch, and S. Kuhr, Nature (London) **471**, 319 (2011).
[56] E. Zohar and B. Reznik, arXiv:1208.1012.
[57] E. Zohar *et al.* (in preparation).
[58] D. Banerjee, M. Bögli, M. Dalmonte, E. Rico, P. Stebler, U.-J. Wiese, and P. Zoller, preceding Letter, Phys. Rev. Lett. **110**, 125303 (2013).
[59] L. Tagliacozzo, A. Celi, P. Orland, and M. Lewenstein, arXiv:1211.2704.