# Measurement-Based Quantum Computation on Symmetry Breaking Thermal States

Keisuke Fujii,[1] Yoshifumi Nakata,[2] Masayuki Ohzeki,[3] and Mio Murao[2,4]

[1]*Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan*
[2]*Department of Physics, Graduate School of Science, University of Tokyo, Tokyo 113-0033, Japan*
[3]*Department of Systems Science, Graduate School of Informatics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan*
[4]*Institute for Nano Quantum Information Electronics, University of Tokyo, Tokyo 153-8505, Japan*
(Received 6 September 2012; published 21 March 2013)

We consider measurement-based quantum computation (MBQC) on thermal states of the interacting cluster Hamiltonian containing interactions between the cluster stabilizers that undergoes thermal phase transitions. We show that the long-range order of the symmetry breaking thermal states below a critical temperature drastically enhances the robustness of MBQC against thermal excitations. Specifically, we show the enhancement in two-dimensional cases and prove that MBQC is topologically protected below the critical temperature in three-dimensional cases. The interacting cluster Hamiltonian allows us to perform MBQC even at a temperature 1 order of magnitude higher than that of the free cluster Hamiltonian.

DOI: 10.1103/PhysRevLett.110.120502 PACS numbers: 03.67.Lx, 03.67.Pp, 05.50.+q, 75.10.Nr

*Introduction.*—Measurement-based quantum computation (MBQC) is a paradigm for quantum computation, where a many-body entangled state is prepared as a universal resource state, and quantum computation can be executed by adaptive single-qubit measurements on it [1]. This paradigm provides a good clue to understand requirements on a system as a resource for universal quantum computation, making a bridge between quantum information science and many-body physics. A central issue in this approach is to specify a many-body system whose ground or low-temperature thermal state can serve as a universal resource for MBQC.

Ground states of several Hamiltonians such as the cluster Hamiltonians [2–4] and valence-bond solid Hamiltonians [5–9] have been found to be universal. At finite temperature, thermal states of several Hamiltonians have been shown to be useful as universal resources by protecting quantum information from errors originating from the thermal excitation by using quantum error correction [10–12]. However, these Hamiltonians do not undergo any physical (thermal or quantum) phase transitions, although they exhibit a transition in computational capability of MBQC by varying temperature. Thus, we address a question whether it is possible to enhance the robustness of MBQC against thermal excitations by introducing a Hamiltonian that undergoes a phase transition. This can strengthen the connection between quantum information science and many-body physics and can provide an approach to understanding the robustness of MBQC in terms of many-body physics.

In this Letter, we show that the robustness of MBQC on thermal states can be enhanced drastically by introducing interactions between the cluster stabilizers. The proposed Hamiltonian—an *interacting* cluster Hamiltonian (iCH)—is transformed into a ferromagnetic Ising Hamiltonian by unitary transformations. Hence, it undergoes a phase transition on two- or higher-dimensional lattices, leading to the symmetry breaking of the thermal states. By virtue of the ferromagnetic-type long-range order of such symmetry breaking states, MBQC becomes robust below the critical temperature. We first demonstrate this on a two-dimensional (2D) lattice and show that the fidelity of MBQC can be drastically improved below the critical temperature due to the long-range order, although it is not sufficiently large at the temperature just below the critical temperature. We further investigate topologically protected MBQC on a three-dimensional (3D) lattice [2,13–19] in order to achieve a quantum computation of arbitrary accuracy at any temperature below the critical temperature. We show that the threshold value for the topologically protected MBQC is exactly equal to the critical temperature of the ferromagnetic Ising Hamiltonian in 3D. Compared to the previous Hamiltonian without the interactions between the cluster stabilizers [2], the temperature required for topologically protected MBQC is relaxed by more than 1 order of magnitude.

*Cluster Hamiltonian.*—The *cluster stabilizer* on a lattice $\mathcal{T}$ is given by $K_i = X_i \bigotimes_{j\in V_i} Z_j$ for each site $i$, where $A_i$ ($A = X, Y, Z$) are the Pauli operators on the $i$th qubit and $V_i$ denotes the set of the vertices that are adjacent to the site $i$ in the lattice $\mathcal{T}$ [1]. The cluster state on the lattice $\mathcal{T}$, $|\Psi_{\mathcal{T}}\rangle$, is defined by the simultaneous eigenstate of all cluster stabilizers $K_i$ with eigenvalue $+1$. The cluster Hamiltonian is defined by using the cluster stabilizers as $H_{\rm fc} = -J\sum_i K_i$ [2] [see Fig. 1(a)], where $J$ is a coupling constant. It is obvious that this Hamiltonian has the cluster state $|\Psi_{\mathcal{T}}\rangle$ as its ground state. By using a unitary transformation $U_{\mathcal{T}}$, the products of CONTROLLED-Z gates on all

 

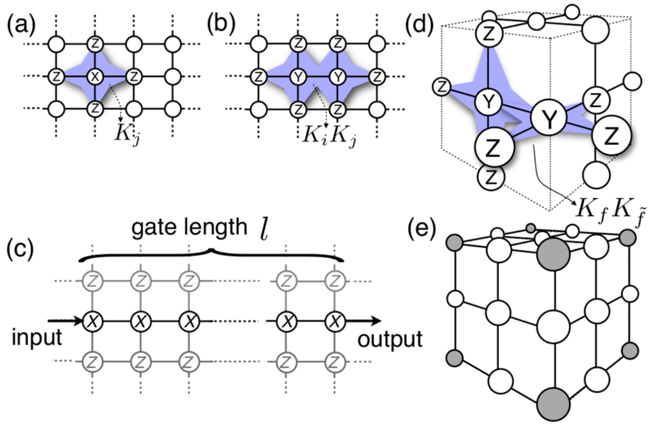


FIG. 1 (color online). (a) The fCH on the square lattice. (b) The iCH on the square lattice. (c) The measurement pattern for the identity and the Hadamard gates of the gate length $l$ on the 2D cluster state. (d) The interacting cluster Hamiltonian on the RHG lattice. (e) The 3D cluster state on the cubic lattice. The gray-shaded qubits are measured in the $Z$ basis to obtain the cluster state on the RHG lattice.

bonds of the lattice $\mathcal{T}$, the Hamiltonian can be transformed into an interaction-free Hamiltonian $U_{\mathcal{T}} H_{\rm fc} U_{\mathcal{T}} = -J\sum_i X_i$. We call this Hamiltonian a *free cluster* Hamiltonian (fCH) hereafter. It should be emphasized that this system does not exhibit any phase transition since the thermal state is equivalent to that for the interaction-free Hamiltonian.

The thermal state of the Hamiltonian $-J\sum_i X_i$ is given by $\prod_i(\mathcal{E}_i|+\rangle\langle+|)$, where $\mathcal{E}_i$ is a map defined by $\mathcal{E}_i\rho = (\rho + e^{-2\beta J}Z_i\rho Z_i)/(1+e^{-2\beta J})$. Thus, the thermal state of the fCH can be calculated as $(\prod_i \mathcal{E}_i)|\Psi_{\mathcal{T}}\rangle\langle\Psi_{\mathcal{T}}|$, where we used the fact that $U_{\mathcal{T}}\mathcal{E}_i\rho U_{\mathcal{T}}^\dagger = \mathcal{E}_i U_{\mathcal{T}}\rho U_{\mathcal{T}}^\dagger$ [2] (see the Supplemental Material for the detailed calculations [20]). The thermal state can be regarded as a cluster state with an independent $Z$ error on each qubit with probability $p_{\beta J} = e^{-2\beta J}/(1+e^{-2\beta J})$. It has been known that it is possible to perform MBQC in a topologically protected way with a 3D cluster state on the so-called Raussendorf-Harrington-Goyal (RHG) lattice [17–19].

Next, we introduce an interacting cluster Hamiltonian (iCH) $H_{\rm ic} = -J\sum_{\langle ij\rangle}K_iK_j$, where the summation runs over all bonds $\langle ij\rangle$ of the lattice $\mathcal{T}$, and thus each cluster stabilizer interacts with its nearest neighbors, as is shown in Fig. 1(b) for the case of the square lattice. This indicates that the iCH generally contains mutually dependent stabilizer operators. The iCH can be transformed to the ferromagnetic Ising Hamiltonian on the lattice $\mathcal{T}$ by $U_{\mathcal{T}}$ as $U_{\mathcal{T}}H_{\rm ic}U_{\mathcal{T}} = -J\sum_{\langle ij\rangle}X_iX_j$, which is denoted by $H_{\rm Ising}$. If the geometrical structure of the lattice is chosen properly, for example, two- or higher-dimensional lattices, phase transitions happen. Although each eigenstate of the iCH $|\Phi\rangle$ is degenerate with $\prod_{i\in\mathcal{T}}Z_i|\Phi\rangle$ due to the symmetry of the iCH [20], we can project them onto a symmetry breaking thermal (SBT) state $\rho_{\rm SBT} = P\rho_{\rm th}P$ by measuring cluster stabilizers, which are denoted by the projective operator $P$. Note that these stabilizer measurements can be implemented by using only single qubit measurements. Then, the SBT state $\rho_{\rm SBT}$ exhibits long-range order. The thermal errors in the Ising model $H_{\rm Ising} = U_{\mathcal{T}}H_{\rm ic}U_{\mathcal{T}}$ can be regarded as strongly correlated $Z$ errors. Since $Z$ errors and $U_{\mathcal{T}}$ are commutable, the thermal state of the iCH has the same $Z$ error distribution as that on the Ising model (see the Supplemental Material for the detailed calculations [20]). MBQC on such SBT states would exploit the robustness of long-range order.

*MBQC on the square lattice iCH.*—Let us first consider MBQC in the iCH on a square lattice where a periodic boundary condition is assumed. Since the iCH is unitarily equivalent to the Ising Hamiltonian, the system undergoes a phase transition at the critical temperature $T_c/J = 2/\ln[1+\sqrt{2}]$ [21]. To check whether the SBT state leads to the robustness of MBQC, we consider performing the identity and Hadamard gates, which are implemented by the $Z$ basis measurements for cutting the cluster state into a line and by the $X$ basis measurements for teleportation-based gates [1,20] [see Fig. 1(c)].

The gate fidelity with the gate length $l$ (identity and Hadamard gates with even and odd $l$, respectively) is given by

$$F(l) = \mathrm{Tr}\left[\frac{I+\prod_{i=1}^{[(l-1)/2]}K_{2i}}{2}\,\frac{I+\prod_{i=1}^{[l/2]}K_{2i-1}}{2}\rho_{\rm th}\right], \quad (1)$$

where $\rho_{\rm th} = e^{-\beta H}/\mathrm{Tr}e^{-\beta H}$ is a thermal state of a given Hamiltonian $H$ [22] (see the Supplemental Material for the detailed derivation [20]). In the case of the iCH, $\rho_{\rm th}$ is replaced by $\rho_{\rm SBT}$. The gate fidelity takes values between $1/4$ and 1, and the minimum gate fidelity $1/4$ implies that the output state is the completely mixed state and the gate operations fail. In the cases of the fCH and the iCH, by applying the unitary transformation $U_{\mathcal{T}}$, the gate fidelity $F(l)$ is expressed in terms of the many-body correlation functions of the interaction-free and the Ising models, respectively. The results for the identity gates, $F(2l)$ with $l = 2, 4, 6$, and 8, are shown in Fig. 2 for the fCH and the iCH (for the gate fidelity of the Hadamard gate, see the Supplemental Material [20]).

For the fCH, the gate fidelities exponentially decrease with an increase of temperature to $1/4$ for any distance $l$. For the iCH, the gate fidelities change differently below or above the critical temperature; namely, the gate fidelities also exhibit a transition at the critical temperature. Above the critical temperature, the gate fidelities are close to $1/4$. In contrast, the SBT state appearing below the critical temperature leads to a dramatic improvement of the gate fidelities by exploiting its long-range order. The temperature required to perform the gate operation reliably for the iCH is much higher than that for the fCH; e.g., the gate fidelity for $l = 8$ is almost 1 if $T \lesssim 1.0J$ for the iCH and $T \lesssim 0.3J$ for the fCH.

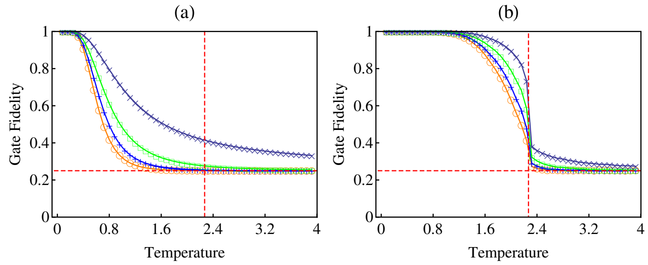


FIG. 2 (color online). The gate fidelities of the identity gates for various distances $l$. (a) shows the gate fidelity for the fCH, $F(2l)$ for $l = 2(\times)$, $l = 4(\Box)$, $l = 6(+)$, and $l = 8(\circ)$. (b) shows the gate fidelity for the iCH, $F(2l)$ for $l = 2(\times)$, $l = 4(\Box)$, $l = 6(+)$, and $l = 8(\circ)$. The vertical dashed line shows the critical temperature for the 2D iCH $T_c/J = 2/\ln[1+\sqrt{2}]$, and the horizontal dashed line shows the minimum gate fidelity $F = 1/4$.

The identity and Hadamard gates are not sufficient for universal MBQC. The fidelities of other gates, such as $Z$ rotations and CONTROLLED-$Z$ gates, are also expressed in terms of the many-body correlation function [22], and we obtain similar results for these gates that form a universal set of gates (see the Supplemental Material [20]). However, even for the iCH, the fidelities just below the critical temperature are not large enough to reliably carry out an arbitrary number of gate operations. This motivates us to consider topologically protected MBQC on the SBT states on a 3D lattice, where fully fault-tolerant and universal quantum computations can be performed.

*Topologically protected MBQC.*—Topologically protected MBQC can be performed with the cluster state on the RHG lattice [17–19]. The RHG lattice is defined by the set of cubes $Q$, the set of faces $F_q$ on each cube $q \in Q$, and the set of edges $E_f$ on each face $f \in F_q$. The qubits are located on each face and edge constituting the 3D cluster state, where the nearest-neighbor stabilizer generators interact with each other, as is shown in Fig. 1(c). Similarly to the original case [17–19], the 3D cluster state is subdivided into three regions, vacuum $V$, defect $D$, and singular qubits $S$ used for the topological protection, performing Clifford gates, and performing an arbitrary single-qubit gate, respectively. In the following, we consider the threshold value for the topological protection in the vacuum region $V$ since it solely determines a threshold of quantum computation [17–19].

In the vacuum region $V$, all qubits are measured in the $X$ basis. Since $\prod_{f\in F_q} K_f = \prod_{f\in F_q} X_f$, the parity of the measurement outcomes on the six face qubits on each unit cell has to be even for the ideal cluster state. Thus, the $Z$ errors on the face qubits, say, error chain $C$, are detected at the boundary $\partial C$, which is called the error syndrome of $C$, since the error chain $C$ anticommutes there with $\prod_{f\in F_q} X_f$. This is also the case for the $Z$ errors on the edge qubits, say, error chain $\bar{C}$, since the edge qubits are the face qubits on the dual lattice. In the case of the fCH on the RHG lattice [2], the error chains $C$ and $\bar{C}$, denoted by $\mathcal{C} \equiv (C, \bar{C})$, are not correlated and can be treated independently. However, in the case of the iCH, the primal and dual error chains of $\mathcal{C}$ are strongly correlated and have to be treated simultaneously.

By using the error syndrome $\partial\mathcal{C}$, we infer the actual locations of the errors. To this end, the probability of a hypothetical error chain $\mathcal{C}'$, which has the same error syndrome $\partial\mathcal{C}$, is calculated to be

$$p(\mathcal{C}'|\partial\mathcal{C}') = \mathcal{N}^{-1}\exp\Big[\beta' J \sum_{\langle f\bar{f}\rangle} u_f^{C'} u_{\bar{f}}^{\bar{C}'}\Big]\Big|_{\partial\mathcal{C}'=\partial\mathcal{C}}, \quad (2)$$

where $\mathcal{N}$ is the normalization factor. We used the knowledge that the errors occur with a ferromagnetic Ising-type distribution (see the Supplemental Material [20]), which is characterized by an inference parameter $\beta'$ independently of the physical inverse temperature $\beta$. The indicator function $u_f^{C'}$ is defined as $u_f^{C'} = -1$ for $f \in C'$ and $u_f^{C'} = 1$ for $f \notin C'$, specifying the location of the errors. Since $\partial C' = \partial C$, we have $\mathcal{C}' = \mathcal{C} + \mathcal{L}$ for trivial loops (cycles) $\mathcal{L} \equiv (L, \bar{L})$, where $L$ is a trivial loop for the lattice and $\bar{L}$ is for the dual lattice, such that $\partial\mathcal{L} = 0$. In order to solve the loop condition, we introduce gauge variables on the edges of primal and dual lattices defined by $P_f \equiv u_f^L = \prod_{e\in E_f}\sigma_e$ and $\bar{P}_{\bar{f}} \equiv u_{\bar{f}}^{\bar{L}} = \prod_{\bar{e}\in \bar{E}_{\bar{f}}}\bar{\sigma}_{\bar{e}}$. In this parametrization, $\prod_{f\in F_q} P_f = 1$, and $\partial L = 0$ is automatically satisfied. As a result, we obtain the Gibbs-Boltzmann distribution $p(\mathcal{C}'|\partial\mathcal{C}) = \mathcal{N}^{-1} e^{-\beta' H_{\mathcal{C}}(\sigma,\bar{\sigma})}$ under a Hamiltonian given by $H_{\mathcal{C}}(\sigma,\bar{\sigma}) = -J\sum_{\langle f\bar{f}\rangle} u_f^C u_{\bar{f}}^{\bar{C}} P_f \bar{P}_{\bar{f}}$, which we call the correlated random-plaquette $\mathbb{Z}_2$ gauge model (cRPGM) [20]. The sign of the two plaquette interactions $u_f^C u_{\bar{f}}^{\bar{C}}$ representing the randomness of the model is determined by the actual error chain $\mathcal{C}$ with the distribution $p(\mathcal{C}) \equiv \mathcal{N}^{-1} e^{\beta J \sum_{\langle f\bar{f}\rangle} u_f^C u_{\bar{f}}^{\bar{C}}}$ parametrized by the physical inverse temperature $\beta$.

Since the threshold value for topologically protected MBQC corresponds to the critical point of the cRPGM [14], our goal is to identify it. Let us consider the optimal case of $\beta' = \beta$, where the actual and hypothetical error distributions are the same. This condition is referred to as the Nishimori line [23] in spin glass theory. In this case, the internal energy is given by

$$[\langle H_{\mathcal{C}}(\sigma,\bar{\sigma})\rangle_{\mathrm{th}}]_{\mathcal{C}} = \sum_{\mathcal{C}} p(\mathcal{C}) \sum_{\{\sigma_e,\bar{\sigma}_{\bar{e}}\}} \frac{H_{\mathcal{C}}(\sigma,\bar{\sigma}) e^{-\beta H_{\mathcal{C}}(\sigma,\bar{\sigma})}}{Z_{\mathcal{C}}(\beta)},$$

where $\langle\cdots\rangle_{\mathrm{th}}$ denotes the thermal average and $Z_{\mathcal{C}}(\beta) = \sum_{\{\sigma_e,\bar{\sigma}_{\bar{e}}\}} e^{-\beta H_{\mathcal{C}}(\sigma,\bar{\sigma})}$ is the partition function. We take the ensemble average of the error distributions $[\cdots]_{\mathcal{C}}$ since the internal energy of the cRPGM has a self-averaging property [24]. With the aid of the gauge symmetry [23], the Hamiltonian $H_{\mathcal{C}}(\sigma,\bar{\sigma})$ is invariant under the following gauge transformations $u_f^C \to u_f^C P'_f$, $\sigma_e \to \sigma_e \sigma'_e$, $u_{\bar{f}}^{\bar{C}} \to u_{\bar{f}}^{\bar{C}} \bar{P}'_{\bar{f}}$, and $\bar{\sigma}_{\bar{e}} \to \bar{\sigma}_{\bar{e}}\bar{\sigma}'_{\bar{e}}$, where $P'_f = \prod_{e\in E_f}\sigma'_e$ and $\bar{P}'_{\bar{f}} = \prod_{\bar{e}\in \bar{E}_{\bar{f}}}\bar{\sigma}'_{\bar{e}}$. On the other hand, these transformations change the distribution $p(\mathcal{C})$ into $\mathcal{N}^{-1}e^{\beta J\sum_{\langle f,\bar{f}\rangle}u_f^C u_{\bar{f}}^{\bar{C}}P'_f\bar{P}'_{\bar{f}}} = \mathcal{N}^{-1}e^{-\beta H_{\mathcal{C}}(\sigma',\bar{\sigma}')} \equiv p'(\mathcal{C})$, which corresponds to the Gibbs-Boltzmann distribution for the cRPGM. Since $\sum_{\{\sigma',\bar{\sigma}'\}}p'(\mathcal{C}) = Z_{\mathcal{C}}(\beta)$, we can mitigate the difficulty in calculating the internal energy of the cRPGM by canceling out $Z_{\mathcal{C}}(\beta)$ as follows:

$$\begin{aligned}[\langle H_{\mathcal{C}}(\sigma,\bar{\sigma})\rangle_{\mathrm{th}}]_{\mathcal{C}} &= \frac{1}{\mathcal{N}}\sum_{\mathcal{C}}\frac{1}{|\mathcal{L}|}\sum_{\{\sigma'_e,\bar{\sigma}'_{\bar{e}}\}}p'(\mathcal{C}) \\ &\quad\times\sum_{\{\sigma_e,\bar{\sigma}_{\bar{e}}\}}\frac{H_{\mathcal{C}}(\sigma,\bar{\sigma})e^{-\beta H_{\mathcal{C}}(\sigma,\bar{\sigma})}}{Z_{\mathcal{C}}(\beta)} \\ &= \frac{1}{\mathcal{N}|\mathcal{L}|}\sum_{\{\sigma_e,\bar{\sigma}_{\bar{e}}\}}\sum_{\mathcal{C}}H_{\mathcal{C}}(\sigma,\bar{\sigma})e^{-\beta H_{\mathcal{C}}(\sigma,\bar{\sigma})} \\ &= \mathcal{N}^{-1}\sum_{\mathcal{C}}H_{\mathrm{Ising}}e^{-\beta H_{\mathrm{Ising}}} = \langle H_{\mathrm{Ising}}\rangle_{\mathrm{th}},\end{aligned}$$

where $|\mathcal{L}|$ is the number of the loop configurations, $H_{\mathrm{Ising}} = -J\sum_{\langle f,\bar{f}\rangle}u_f^C u_{\bar{f}}^{\bar{C}}$, and $\mathcal{N}$ is defined in Eq. (2) as the partition function of the Ising model. For the transformation from the third to the fourth lines, we take the summation $\sum_{\sigma_e,\bar{\sigma}_{\bar{e}}} = |\mathcal{L}|$ by using the fact that $u_f^C P_f = u_f^{C+L}$ and $u_{\bar{f}}^{\bar{C}}\bar{P}_{\bar{f}} = u_{\bar{f}}^{\bar{C}+\bar{L}}$, with trivial loops $\mathcal{L}$ and $\sum_{\mathcal{C}} = \sum_{\mathcal{C}+\mathcal{L}}$. That is, the gauge variables $P_f$ and $\bar{P}_{\bar{f}}$ are completely absorbed into the coupling constant $u_f^C u_{\bar{f}}^{\bar{C}}$ by changing the variables $\mathcal{C} \to \mathcal{C} + \mathcal{L}$. Thus, the internal energy of cRPGM is equivalent to that of the Ising model without any randomness.

In the Ising model on the RHG lattice, the internal energy has a nonanalytical point at $T_c = 2.8$, which is evaluated by the exchange Monte Carlo simulation [25]. Therefore, we can conclude that the internal energy of the cRPGM along the Nishimori line also has a nonanalytical point at $T_c = 2.8$, which is the phase boundary of the Higgs (ordered) and confinement (disordered) phases [14–16]. In the Higgs phase, the loop configurations $\mathcal{L}$ of large perimeters are exponentially suppressed. Thus, the logical error probability, which is characterized by the loop configurations of nontrivial topology, is decreased exponentially by increasing the size of the system (see the Supplemental Material [20] for the decoding methods). That is, the transition point of the performance in topologically protected MBQC on the SBT states is exactly determined by the critical temperature of the phase transition in the underlying physical system.

The cluster state on the RHG lattice can be also obtained from other lattices such as simple cubic (SC), face-centered cubic, and close-packed hexagonal lattices by measuring appropriate qubits in the $Z$ basis, as is shown in Fig. 1(d) in the case of the SC lattice. Since the thermal errors commute with the $Z$ basis measurements in our model, they do not induce any additional errors. Also, in these cases, by using the gauge transformation, we can show that the thresholds for topologically protected MBQC are again given by the critical temperatures of the iCHs on those lattices. The critical temperatures of the Ising models on the SC, close-packed hexagonal, and face-centered cubic lattices have been calculated numerically as $T_c = 4.5$, 9.3, and 9.8, respectively [26], which are higher than $T_c = 2.8$ for the RHG lattice, since each site interacts with more neighboring sites. In comparison with $T_c = 0.59$ for the fCH, the iCHs with the long-range order relax the temperature required for topologically protected MBQC by more than 1 order of magnitude. In the fCHs, the lattice structures do not change the threshold value for the topological protection since the thermal errors occur independently for each qubit. Contrarily, in the iCHs, the underlying lattice structures take a very important role in robustness against the thermal excitation by making use of physical cooperative phenomena.

*Conclusions and discussions.*—We have demonstrated that the physical cooperative phenomena of a system can help MBQC on the system even at finite temperature. We have first shown that, in a square lattice, the gate fidelities of the identity gates for the iCH are drastically improved compared to those for the fCH below the critical temperature. It has been also shown that the fidelities are not sufficiently large for performing MBQC reliably at the temperature just below the critical temperature even for the iCH. In the 3D cases, MBQC on the thermal states are topologically protected below the critical temperatures of the underlying physical system, which allows us to perform MBQC on the SBT states even at much higher temperatures than the models without physical cooperative phenomena. A promising way to design these many-body interactions used in both fCH and iCH is the stabilizer pumping scheme [27–30] (see the Supplemental Material for a detailed discussion [20]). Although achieving larger many-body interactions requires more unitary operations in the scheme, the required temperatures for performing topologically protected MBQC is significantly relaxed for the iCH.

In the present work, we have considered only the Ising-type interaction in the stabilizer Hamiltonian. We can also construct the iCHs, which are unitarily equivalent to other spin models such as the Potts, $XY$, and Heisenberg models. It is an interesting future work to study the relation between the ordered phase and quantum information tasks in such models. This will open up a new approach to make use of physical cooperative phenomena for quantum information processing.

The authors thank H. Nishimori and S. Tanaka for fruitful discussions. This work was supported by MEXT, Japan (Project for Developing Innovation Systems, Grant-in-Aid for Scientific Research on Innovative Areas No. 20104003, and Grant-in-Aid for Young Scientists (B) No. 24740263), and by JSPS (Grants No. 222812, No. 23540463, and No. 23240001).

[1] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001); R. Raussendorf, D. E. Browne, and H. J. Briegel, Phys. Rev. A **68**, 022312 (2003).
[2] R. Raussendorf, S. Bravyi, and J. Harrington, Phys. Rev. A **71**, 062313 (2005).
[3] F. Verstraete and J. I. Cirac, Phys. Rev. A **70**, 060302(R) (2004).
[4] S. D. Bartlett and T. Rudolph, Phys. Rev. A **74**, 040302(R) (2006).
[5] G. K. Brennen and A. Miyake, Phys. Rev. Lett. **101**, 010502 (2008).
[6] X. Chen, B. Zeng, Z.-C. Gu, B. Yoshida, and I. L. Chuang, Phys. Rev. Lett. **102**, 220501 (2009).
[7] J. Cai, A. Miyake, W. Dür, and H. J. Briegel, Phys. Rev. A **82**, 052309 (2010).
[8] T.-C. Wei, I. Affleck, and R. Raussendorf, Phys. Rev. Lett. **106**, 070501 (2011).
[9] A. Miyake, Ann. Phys. (Amsterdam) **326**, 1656 (2011).
[10] S. D. Barrett, S. D. Bartlett, A. C. Doherty, D. Jennings, and T. Rudolph, Phys. Rev. A **80**, 062328 (2009).
[11] Y. Li, D. E. Browne, L. C. Kwek, R. Raussendorf, and T.-C. Wei, Phys. Rev. Lett. **107**, 060501 (2011).
[12] K. Fujii and T. Morimae, Phys. Rev. A **85**, 010304 (2012).
[13] A. Yu. Kitaev, Ann. Phys. (Amsterdam) **303**, 2 (2003).
[14] E. Dennis, A. Yu. Kitaev, A. Landahl, and J. Preskill, J. Math. Phys. (N.Y.) **43**, 4452 (2002).
[15] C. Wang, J. Harrington, and J. Preskill, Ann. Phys. (Amsterdam) **303**, 31 (2003).
[16] T. Ohno, G. Arakawa, I. Ichinose, and T. Matsui, Nucl. Phys. **B697**, 462 (2004).
[17] R. Raussendorf, J. Harrington, and K. Goyal, Ann. Phys. (Amsterdam) **321**, 2242 (2006).
[18] R. Raussendorf and J. Harrington, Phys. Rev. Lett. **98**, 190504 (2007).
[19] R. Raussendorf, J. Harrington, and K. Goyal, New J. Phys. **9**, 199 (2007).
[20] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.110.120502 for detailed calculations.
[21] L. Onsager, Phys. Rev. **65**, 117 (1944).
[22] A. C. Doherty and S. D. Bartlett, Phys. Rev. Lett. **103**, 020506 (2009).
[23] H. Nishimori, Prog. Theor. Phys. **66**, 1169 (1981).
[24] H. Nishimori, *Statistical Spin Glasses and Information Processing: An Introduction* (Oxford University Press, New York, 2001).
[25] K. Hukushima and K. Nemoto, J. Phys. Soc. Jpn. **65**, 1604 (1996).
[26] A. N. C. de Magalhaes and C. Tsallis, J. Phys. IV **42**, 1515 (1981).
[27] H. Weimer, M. Müller, I. Lesanovsky, P. Zoller, and H. P. Büchler, Nat. Phys. **6**, 382 (2010).
[28] J. T. Barreiro, M. Müller, P. Schindler, D. Nigg, T. Monz, M. Chwalla, M. Hennrich, C. F. Roos, P. Zoller, and R. Blatt, Nature (London) **470**, 486 (2011).
[29] B. P. Lanyon *et al.*, Science **334**, 57 (2011).
[30] M. Müller, K Hammerer, Y L Zhou, C F Roos, and P Zoller, New J. Phys. **13**, 085007 (2011).