

**Wilczek Reply:** Professor Bruno [1] identifies a lower-energy alternative to the soliton motion calculated in Ref. [2] in a specific case, but this does not justify a general conclusion.

The immediate issue, for the flux mechanism of time crystals, is the effect of magnetic flux on the behavior of a quantum many-body system near its ground state. (For convenience, I will speak of magnetic flux, charge, and so forth, as familiar labels for parameters in an effective Lagrangian, which could have alternative interpretations.) It is useful first to step back from details and consider some relevant structural features of the problem. For concreteness, consider an  $N$ -body system of charged particles described by coordinates  $x_j$  and translation-invariant interactions; we can introduce the center of mass  $X \equiv \frac{1}{N} \sum_{j=1}^N x_j$  and relative coordinates  $y_k \equiv x_k - x_{k+1}$ ,  $1 \leq k \leq N-1$ .

Then, the Hamiltonian decomposes into a center-of-mass piece depending only on  $X$  and a relative piece depending only on the  $y_k$ . One can therefore generate solutions by factorizing, as presented in Ref. [2]. The low-energy dynamics of  $X$  can nevertheless depend in a nontrivial way, in other solutions. To appreciate how, from a mathematical perspective, let us write the ground state wave function of the relative coordinate Hamiltonian for a fixed value of the center of mass  $X_0$ , as  $\Phi_r(x_j; X_0)$ . Although the action of the Hamiltonian on  $\psi$  in superpositions  $\Psi_v(x_j) \equiv \int dX_0 \psi(X_0) \Phi_r(x_j; X_0)$  is essentially that of a free particle, the center-of-mass part of the Hamiltonian can also act nontrivially on  $\Phi_r(x_j; X_0)$ . Alternatively stated, the normalized overlap integral  $\int d^N x_j \Psi_v^*(x_j) \Psi_v(x_j)$  may not reproduce the standard reduced overlap  $\int dX \psi^*(X) \psi(X)$  but instead induces a different measure. For example, in the extremely inhomogeneous “collapse” case  $\Phi_r(x_k; X_0) = \prod_{k=1}^N \delta^{1/2}(x_k - X_0)$ , the overlaps  $\int d^N x_j \Phi_r^*(x_j; X'_0) \times \Phi_r(x_j; X_0)$  vanish drastically for  $X'_0 \neq X_0$ , so there is no communication among different  $X$  values. In that case, any  $\psi(X)$  wave function preserves its form in time. (The square root of the delta function stands in for some regulated wave function whose square is normalized.) At the other extreme, the factorized case  $\Phi_r(x_j; X_0) = \delta^{1/2}(X - X_0) f(y_k)$ , where  $\int d^{N-1} y f(y_k)^2$  is finite, and reflects the minimal dependence on  $X_0$  through overall normalization. It allows  $\psi(X)$  to behave like an ordinary—simple, but nontrivial—free-particle wave function.

These very general considerations underlie, for example, the microscopic description of ordinary crystals and, more generally, spontaneous symmetry breaking. In that case,  $\Phi_r$  reflects the periodicity of the crystal, and in the large  $N$  limit the reduced overlaps vanish unless  $X'$  and  $X$  differ by a period. Then, formal superpositions  $\psi(X)$  that include incommensurate  $X$  values correspond to “Schrödinger cat” states; in that sense, the center-of-mass variable  $X$  is localized. Time crystals in our sense expand the usual

framework by contemplating the possibility that in some situations the dynamics might favor the nontrivial time dependence  $X(t)$  of a localized  $X$  or of a long-lived (in the limit  $N \rightarrow \infty$ ) wave packet induced by an infinitesimal perturbation.

In the cases considered in Refs. [2,3], we isolate angular variables  $\phi_k$  for the angles of the  $x_k$  around a preferred axis, along which there are a uniform flux and corresponding angles  $\theta_k$ ,  $N\Theta$  for  $y_k$ ,  $X$ , and suppose that the corresponding radial variables are frozen at  $R$ . (Note that the total  $N$  for the entire system can correspond to a much smaller effective  $n$ , governing the charge and angular dynamics, in the transverse section.) The Hamiltonian for the relative coordinates does not contain the flux, and the Hamiltonian dependence on  $\Theta$  is quasifree. Nevertheless, qualitatively different behaviors for the dynamics of  $\Theta$  can occur through the dependence on overlaps. In the limiting case of collapse, the flux has no effect, as Bruno correctly points out. In the factorized, or near-homogeneous, case, it can lead to approximate free-particle dynamics, as I originally contemplated. Even here, there is another possible subtlety—it may be that the relative wave functions adjust so as to drive, indirectly through modified overlaps, more favorable energetics for the center of mass. These qualitative considerations suggest that near-homogeneous systems with significant phase rigidity, e.g., supercurrents “marked” along the lines suggested in Ref. [3] or as realized in the annular Josephson geometry [4], are especially promising candidate arenas for time crystal behavior. As with all cases of spontaneous symmetry breaking, nonuniversal quantitative energetics must be considered on a case-by-case basis.

Finally, let me emphasize that none of this affects the classical time crystal models discussed in Ref. [5] or the quantization of those models discussed in Ref. [6] and add that Ref. [7] is a useful complement to this note.

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