Comment on "Quantum Time Crystals"

In a recent Letter [1], Wilczek proposes the existence of a new state of matter, "quantum time crystals," defined as systems which, in their quantum-mechanical ground state, exhibit periodic oscillations of some physical observable. The proposed model consists of (discernible) particles on an Aharonov-Bohm (AB) ring, with a contact attractive interaction. Using mean-field theory, the problem reduces to the nonlinear Schrödinger equation (NLSE), $i\partial_t\psi = [\frac{1}{2}(-i\partial_\phi - \alpha)^2 - \lambda|\psi|^2]\psi$, with periodic boundary condition $\psi(\phi + 2\pi) = \psi(\phi)$, and $\int_0^{2\pi} d\phi|\psi|^2 = 1$. For zero AB flux ($\alpha = 0$), the particles form a lump (bright soliton) for $\lambda \geq \frac{\pi}{2}$. For nonzero flux α, Wilczek constructs a solution of the NLSE in which the zero-flux soliton rotates with velocity $\omega = \alpha$ (so that the apparent flux vanishes in the rotating frame); the flux-induced energy change is $\Delta\epsilon = \frac{\alpha^2}{2}$ per particle. (I restrict here the discussion to $|\alpha| \leq \frac{1}{2}$, since all physical properties are periodic in α, with period 1 [2].) This is interpreted as being due to Faraday's electromotive torque, which accelerates the lump to velocity $\omega = \alpha$ as the flux is ramped up from zero to α. The gained energy $\Delta\epsilon$ is just the corresponding rotational kinetic energy. Wilczek claims without further justification that this rotating-soliton solution is the ground state and concludes that his model thus constitutes a "quantum time crystal."

Furthermore, Wilczek's result leads to paradoxical (unphysical) consequences. (i) In the large coupling limit ($\lambda \to +\infty$), the soliton width shrinks to zero like λ^{-1}, and the wave function amplitude near the antipode of the soliton shrinks exponentially ($|\psi| \sim \sqrt{\lambda}e^{-\lambda\pi/2}$). The sensitivity to the AB flux α should also be exponentially small (in particular, $\Delta\epsilon$ should be exponentially small, too), and the dynamics of a classical lump (which is completely insensitive to the AB flux and has a static ground state) should be recovered for $\lambda \to +\infty$, in striking contrast with Wilczek's result. (ii) When coupled to the environment (e.g., the electromagnetic field, if the particles carry some electric charge), the rotating lump would radiate energy while being in its ground state, thereby violating the principle of energy conservation. Wilczek's suggestion that the coupling to the environment could be reduced by using higher multipoles or by placing the system in a cavity does not address the paradox convincingly.

These remarks strongly suggest that Wilczek's rotating soliton is *not* the ground state and that the true ground state is actually a stationary state, as I show below. The solution of the NLSE for arbitrary flux [3] is too lengthy and technical to fit in this Comment; thus, I shall give here only the solution for $\alpha = \frac{1}{2}$, which is sufficient to disprove Wilczek's claim. Noticing that the flux α can be gauged away from the NLSE by the transformation $\psi(\phi) = e^{i\alpha\phi}\tilde{\psi}(\phi)$ results in the twisted boundary condition $\tilde{\psi}(\phi + 2\pi) = e^{-i2\pi\alpha}\tilde{\psi}(\phi)$. So, for $\alpha = \frac{1}{2}$, one simply has to solve the NLSE with $\alpha \equiv 0$ and an antiperiodic boundary condition. The correct ground state has the following stationary wave function: $\tilde{\psi}(\phi) = \frac{kK}{\pi\sqrt{\lambda}}\,\mathrm{cn}(\frac{\phi K}{\pi}, k)$. $K \equiv K(k)$ and $E \equiv E(k)$ are the complete elliptic integrals of the first and second kind, and $\mathrm{cn}(u, k)$ is a Jacobi elliptic function [4]; the elliptic modulus k satisfies $[E - (1 - k^2)K]K = \frac{\pi\lambda}{2}$. The chemical potential is $\mu = \frac{K^2}{\pi^2}(\frac{1}{2} - k^2)$, and the total energy per particle is $\epsilon \equiv \mu + \frac{\lambda}{2}\int_0^{2\pi} d\phi|\psi|^4 = -\frac{K^2[(2k^2-1)E-(1-k^2)(3k^2-1)K]}{6\pi^2[E-(1-k^2)K]}$ [5]. Solving explicitly these equations confirms that the present state has a lower energy than Wilczek's state. For $\lambda \to +\infty$, fully analytical results can be obtained for any value of α, yielding the simple asymptotic result $\Delta\epsilon = -3[1 - \cos(2\pi\alpha)]\lambda^2 e^{-\pi\lambda}$, which is negative because the lump is narrower for $\alpha = \frac{1}{2}$ than for $\alpha = 0$, leading to more effective attractive coupling, and is thus considerably lower than Wilczek's result ($\Delta\epsilon = \frac{\alpha^2}{2}$) and does not lead to any unphysical paradox.

In the light of the above discussion, it seems that the very existence of "quantum time crystals" remains highly speculative.

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[4] *NIST Handbook of Mathematical Functions*, edited by F. W. J. Olver *et al.* (Cambridge University Press, Cambridge, England, 2010).
[5] Wilczek's omission of the double-counting correction for the interaction energy in Ref. [1] has been corrected here.

