Quantum Law of Rare Events for Systems with Bosonic Symmetry

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In classical physics, the joint probability of a number of individually rare independent events is given by the Poisson distribution. It describes, for example, the unidirectional transfer of a population between the densely and sparsely populated states of a classical two-state system. We derive a quantum version of the law for a large number of noninteracting systems (particles) obeying Bose-Einstein statistics. The classical law is significantly modified by quantum interference, which allows, among other effects, for the counterflow of particles back into the densely populated state. The suggested observation of this classically forbidden counterflow effect can be achieved with modern laser-based techniques used for manipulating and trapping cold atoms.

In classical physics and statistics, the probability for a number of individually rare events is universally given by the Poisson distribution (see, for instance, Ref. [1]). For example, it is obeyed by a classical gas escaping into an empty space through a penetrable membrane. With the number of atoms N large, and the transition probability made proportionally small, the number of escaped atoms is governed by the Poisson law, with the number of atoms recaptured into the original reservoir vanishing as $N \to \infty$. The validity of the Poisson distribution depends on that one can, in principle, know not only how many but also which of the atoms have escaped. Quantum mechanics offers a different possibility: for identical particles one is allowed to know only the number of the escapees, and not their identities. While it is well known that both Fermi-Dirac and Bose-Einstein symmetries of a wave function may lead to non-Poissonian effects in the full counting statistics of otherwise independent particles [2–8], the failure of the Poisson law in the limit of rare events is less obvious. The subject of this Letter is the general question of what replaces the classical Poisson law in a quantum situation where only the total number of rare events, but not their individual details, can be observed.

We specify to the case of many noninteracting bosons, each of which may occupy one of the two available states. Such systems are also of practical interest, e.g., for their potential applications as detectors. For example, if the transmission amplitude between two connected cavities is influenced by a passing particle, the change observed in the photonic current will announce the particle's arrival. In a similar way, the atomic current of a weakly interacting Bose-Einstein condensate (BEC) trapped in a double- or multiwell potential (see Fig. 1) can be used to gain information about the state of a qubit coupled to the BEC [9–11]. A detailed analysis of the work of such hybrid bosonic devices must take into account the manner in which, and how frequently, the bosonic subsystem is observed, and will be given elsewhere.

We note that the problem is fundamentally different from that of the frequently studied coined quantum walk [12], where interference between virtual paths available to a single particle modifies the classical Gaussian distribution. In our case, modification of the classical law is a many-body effect, specific to the Bose-Einstein statistics.

We start by constructing transition amplitudes for a single particle, which can occupy one of the two levels in an asymmetric double well potential. In terms of the Pauli matrices, the Hamiltonian reads

$$\hat{H} = \epsilon\sigma_z + \xi\sigma_x + \eta\sigma_y, \tag{1}$$

where the spin states $|1\rangle$ and $|2\rangle$, aligned up and down the z axis, represent an atom in the right and left state, respectively, ϵ is the difference between the energies of the states, and ξ and η together define the tunneling matrix element, $T = \xi + i\eta$. For the evolution operator $\hat{U}(t) = \exp(-i\hat{H}t)$ we have

$$\hat{U}(t) = I\cos(\omega t) - i\omega^{-1}[\epsilon\sigma_z + \xi\sigma_x + \eta\sigma_y]\sin(\omega t), \tag{2}$$
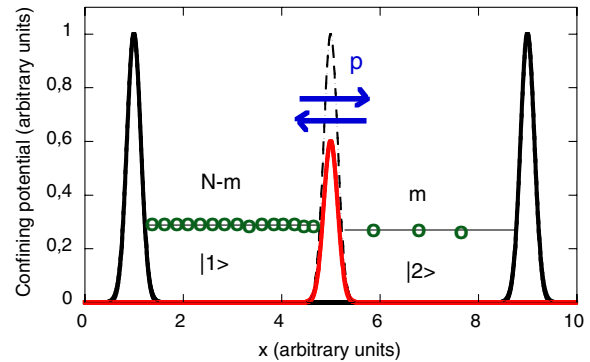

FIG. 1 (color online). Double-well trap containing N atoms. The central barrier is lowered to allow tunneling between the states $|1\rangle$ and $|2\rangle$.

with $\omega = \sqrt{\epsilon^2 + \xi^2 + \eta^2}$, and its matrix elements are conveniently written as (the star indicates the complex conjugate)

$$U_{11} = \sqrt{1 - p}\, \exp(i\alpha) = U_{22}^*,$$
$$U_{12} = \sqrt{p}\, \exp(i\beta) = -U_{21}^*, \qquad (3)$$

where, with our choice of the basis, $p(t) = (\xi^2 + \eta^2)\sin^2(\omega t)/\omega^2$ is the one-particle transition probability, $\alpha(t) = -\arctan[\epsilon \tan(\omega t)/\omega]$, and $\beta = -\pi/2$.

For a total of N particles, we wish to evaluate the transition probabilities $p_{m'\leftarrow m}^N(t)$ for starting with m particles in the state $|1\rangle$ and ending, after a time t, with m' particles in the same state. It is instructive to begin with a brief discussion of the case where all particles are considered distinguishable. The problem is equivalent to a classical N-coin one: given that each coin changes its state with a probability p, and that m coins are initially oriented heads up, what is the probability to have m' heads up after each coin has been tossed once? The result can be achieved by moving ν coins from tails to heads, and μ coins from heads to tails, provided $\nu - \mu = m' - m$. Summing the corresponding probabilities, while taking into account the number of ways to choose the coins which change their state, yields

$$p_{m'\leftarrow m}^N = \sum_{\mu=0}^{m} \sum_{\nu=0}^{N-m} C_\mu^m C_\nu^{N-m}$$
$$\times\, p^{\nu+\mu}(1 - p)^{N-\mu-\nu}\delta_{\nu-\mu,m'-m}, \qquad (4)$$

where $C_l^k \equiv \frac{k!}{l!(k-l)!}$ is the binomial coefficient, and δ_{mn} is the Kronecker delta. Depending on N, m, and m', the sum in Eq. (4) may contain different numbers of terms, corresponding to the number of 'pathways' connecting the initial and final states (filled dots in Fig. 2). In the rare events (RE) limit
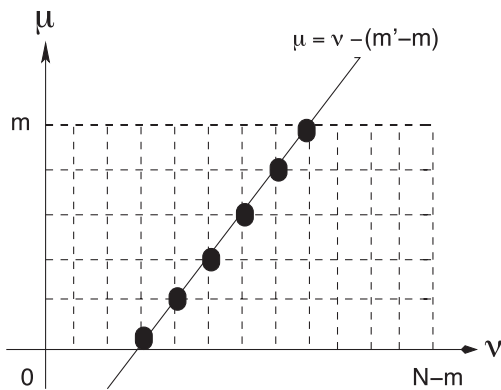

FIG. 2. A diagram showing the region of summation in Eqs. (4) and (9) (filled dots). Each dot contributes $C_\mu^m C_\nu^{N-m} p^{\nu+\mu}(1 - p)^{N-\mu-\nu}$ for distinguishable particles, and $C_\mu^m C_\nu^{N-m}(-1)^\mu U_{12}^{\nu+\mu} U_{11}^{m-\mu} U_{22}^{N-m-\mu}$ for identical bosons.

$$N \to \infty, \qquad p \to w/N, \qquad (5)$$

it is sufficient to retain only the leading $\mu = 0$ terms (the lowest dot in the diagram in Fig. 2) in Eq. (4). Recalling that $C_m^N \sim N^m/m!$ for large N, we obtain the expected Poisson distribution,

$$\lim_{N\to\infty} p_{m'\leftarrow m}^N = \begin{cases} w^q \exp(-w)/q! & q \equiv m' - m \geq 0 \\ 0 & q < 0, \end{cases} \qquad (6)$$

with intuitively appealing properties. Indeed, in the case of a symmetric trap, $\epsilon = 0$, reducing the transition probability in Eq. (5) also makes the Rabi period $2\pi/\omega$ after which the system must return to its initial state extremely large. Now, for all $t \ll 2\pi/\omega$, the evolution can be considered approximately irreversible, with the number of particles q escaping into the right trap independent of the number of particles m already there. The low probability of each individual event and the much lower population in the right well make recrossings from right to left extemely unlikely (see Fig. 3). In particular, after detecting m particles in the right well, one never finds it empty again, as the probability $p_{0\leftarrow m}^N$ (the left upper corner in the diagram in Fig. 2) vanishes as $(w/N)^m \exp(-w)$. One might expect a similar argument to be also valid should distinct particles be replaced with noninteracting bosons. Next we will show that this is not the case.

For identical bosons we have a quantum version of the N-coin problem: after a toss each coin changes its state from $|i\rangle$ to $|j\rangle$, $i, j = 1, 2$ with the probability amplitude U_{ji}, and we must sum amplitudes rather than probabilities over all pathways leading to the same final state. The state of the system with any m coins displaying heads is given by a symmetrized wave function
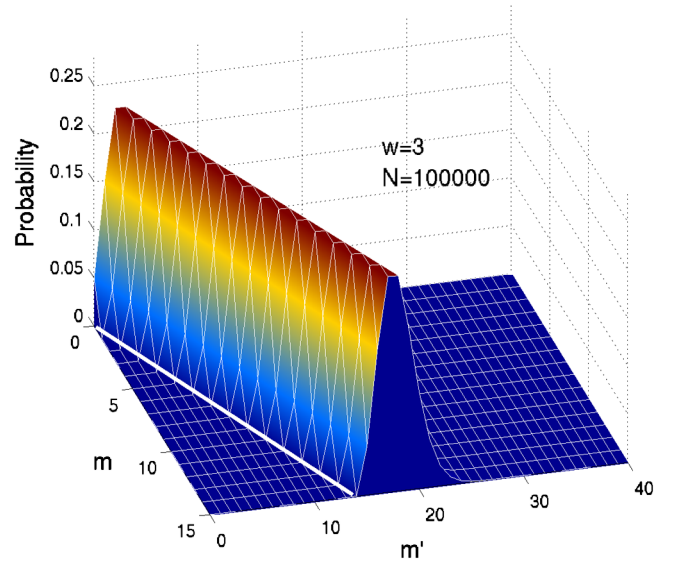

FIG. 3 (color online). Poissonian probability $p_{m'\leftarrow m}^N$ for distinguishable particles, as given by Eq. (4), for $N = 10^5$ and $w = 3$. The white line marks $m = m'$.

$$|m, N\rangle = (C_m^N)^{-1/2} \sum \prod_{j=1}^{N} |i_j\rangle_j, \qquad i_j = 1, 2, \quad (7)$$

where $|i\rangle_j$, $i = 1, 2$ indicates the state of jth particle. The sum is over C_m^N different ways to ascribe to m of the N indices i_j the value of 2, and to the remaining $N - m$ ones the value of 1. After all the coins are tossed once each individual term in the sum of Eq. (7) contributes to the amplitude to have m' heads up a quantity

$$f(m' \leftarrow m, N) = (C_m^N)^{-1/2}(C_{m'}^N)^{-1/2}$$
$$\times \sum_{\mu=0}^{m} \sum_{\nu=0}^{N-m} C_\mu^m C_\nu^{N-m} U_{12}^\nu U_{21}^\mu$$
$$\times U_{11}^{m-\mu} U_{22}^{N-m-\nu} \delta_{m'-m,\nu-\mu}, \quad (8)$$

with the region of summation illustrated in Fig. 2. Since Eq. (7) contains C_m^N such terms, the probability to have m' heads up after the toss is $P_{m'\leftarrow m}^N = (C_m^N)^2 |f(m' \leftarrow m, N)|^2$, $m, m' = 0, 1, \ldots, N$, which, with the help of Eq. (3), can be expressed in terms of the Jacobi polynomials $\mathcal{P}_n^{(\alpha,\beta)}(x)$ [13–15]

$$P_{m'\leftarrow m}^N = \frac{m!(N-m)!}{m'!(N-m')!} p^{m'-m}(1-p)^{N-m'+m}$$
$$\times |\mathcal{P}_m^{(N-m'-m,m'-m)}(2p-1)|^2. \quad (9)$$

As in the case of distinguishable particles, $P_{m'\leftarrow m}^N$ depends only on the one-particle transition probability p, and not on the phases α and β of the matrix elements of U_{ij} in Eq. (3) [16]. We note also that in the special case of tunneling into an initially empty well, $m = 0$, there is only one pathway (moving exactly m particles from left to right), and the transition probabilities for distinguishable particles and identical bosons coincide,

$$P_{m'\leftarrow 0}^N = p_{m'\leftarrow 0}^N = C_{m'}^N p^{m'}(1-p)^{N-m'}, \quad (10)$$

as was pointed out earlier in Refs. [10,11].

More interesting, however, are the transitions affected by the interference effects which, as we will demonstrate, persist even in the RE limit [Eq. (5)]. Indeed, since the sum in Eq. (8) contains $\sqrt{p}$ rather than p, the restriction to only $\mu = 0$ terms is no longer justified. Thus, after taking the limit [Eq. (5)], we have ($q = m' - m$)

$$\lim_{N\to\infty} P_{m'\leftarrow m}^N = w^q \exp(-w)$$
$$\times \left| \sum_{\mu=\max[0,-q]}^{m} \frac{\sqrt{m'!m!}(-w)^\mu}{\mu!(m-\mu)!(q+\mu)!} \right|^2. \quad (11)$$

Equation (11), which is our central result [17], replaces the classical Poisson law [Eq. (6)] for noninteracting identical bosons. Some of its properties are counterintuitive, as is shown in Fig. 4. Firstly, unlike the Poisson distribution [Eq. (6)], $P_{m'm}^N$ of Eq. (11) is highly structured, as a result
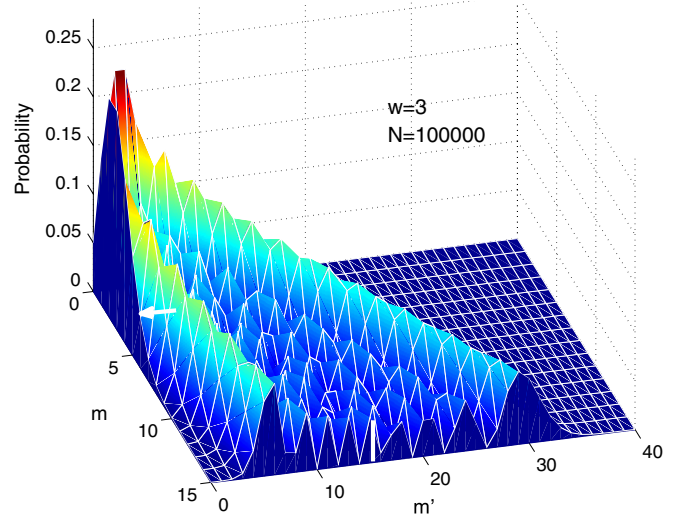

FIG. 4 (color online). Non-Poissonian probability $P_{m'\leftarrow m}^N$ for identical bosons, as given by Eq. (9), for $N = 10^5$ and $w = 3$. The section of the surface indicated by the arrow corresponds to $P_{0\leftarrow m}^N$ also shown in Fig. 5.

of the interference between the pathways. Secondly, it allows for total or partial recapture of the few particles initially held in the right well back into the densely populated left well, contrary to the simple argument based on the improbability of such an event.

The probability for all m bosons to cross into the left well, $P_{0\leftarrow m}^N$, contains only one term in the sum [Eq. (11)] [($\mu = m$, $\nu = 0$) in the diagram in Fig. 2],

$$P_{0\leftarrow m}^N = w^m \exp(-w)/m!. \quad (12)$$

As a function of the number of recaptured atoms m, it is the Poisson distribution shown in Fig. 5 (apparently so ubiquitous that after having been evicted from one part of this Letter, it immediately reappears in another, albeit in a different context). The recapture process exhibits certain resonancelike behavior. The number of particles most likely to be readmitted to the left well, $m \approx w$, equals
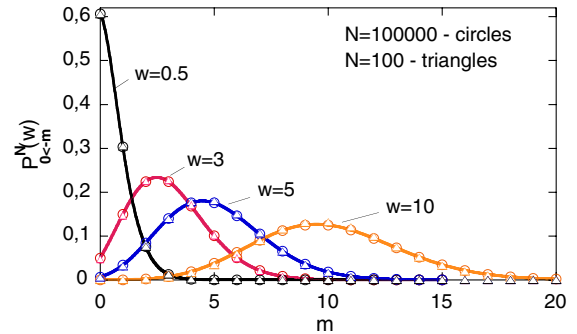

FIG. 5 (color online). The probability $P_{0\leftarrow m}^N$ in Eq. (9) that all m particles initially in the right well, $m \ll N$, $N \gg 1$, would cross to the left, leaving the right well empty. The solid lines are the corresponding Poisson distributions (12).
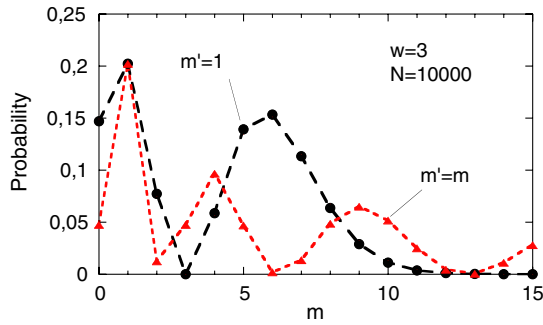
FIG. 6 (color online).   The probabilities $P_{1 \leftarrow m}^{N}$ to end up with just one atom in the right well (circles, long dashed), and $P_{m \leftarrow m}^{N}$, to leave the population of the right well unchanged (triangles, dashed), as functions of m.

the mean number of distinguishable particles crossing into the right well under the same conditions. For $m > 0$, there are two interfering scenarios leading to only one particle being left in the right well [points ($\mu = m - 1$, $\nu = 0$) and ($\mu = m$, $\nu = 1$) in Fig. 2], and the corresponding probability is bimodal as shown in Fig. 4. Similarly, the probability $P_{m \leftarrow m}^{N}$ to retain the same number of atoms in the right well builds up from $m + 1$ interfering terms and also shows an oscillatory pattern (see Fig. 6).

To conclude, we suggest a simple experimental setup to test the recapture property of the bosonic distribution. Using the available laser technology [3] one can create a quasi-one-dimensional box trap with two strong endcap lasers providing the potential walls shown in Fig. 1. The box is divided in two by adding a third laser in the middle, and the left well is populated with a large number N of weakly interacting atoms, while, say, three atoms are introduced into the right well. Following this, the middle laser beam is slightly weakened to allow the transfer of atoms between the wells. It is restored after a time τ, such that $p(\tau) \approx \xi^2 \tau^2 \approx 3/N$, and the number of atoms in the right well is measured, e.g., by a technique described in Ref. [18]. Then, no matter how large N is, the well is found empty with the probability $P_{0 \leftarrow m}^{N} \approx 0.23$ (cf. Fig. 5), i.e., in just under a quarter of all cases.

In summary, it is shown that for non- or weakly interacting bosons quantum interference between different scenarios leading to the same final state modifies the classical Poisson law of rare events, and leads to significant observable effects not present in classical statistics.

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