Direct Statistical Simulation of Out-of-Equilibrium Jets

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We present direct statistical simulation of jet formation on a β plane, solving for the statistics of a fluid flow via an expansion in cumulants. Here we compare an expansion truncated at second order (CE2) to statistics accumulated by direct numerical simulations. We show that, for jets near equilibrium, CE2 is capable of reproducing the jet structure (although some differences remain in the second cumulant). However, as the degree of departure from equilibrium is increased (as measured by the zonostrophy parameter), the jets meander more and CE2 becomes less accurate. We discuss a possible remedy by inclusion of higher cumulants.

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Jets are relatively narrow bands of fast-flowing fluid moving coherently in one direction. They are ubiquitous in nature, found in Earth's oceans and atmosphere, the outer layers of gas giant planets, the interior of stars, and laboratory experiments with fluids and plasmas [1]. Jets play an important role for these fluids, and it is therefore important to understand the mechanism(s) that govern their formation. Sometimes jets are driven by energy input at small spatial scales; the question then is how this energy is transferred into large-scale coherent motion. Two competing mechanisms have been proposed, both of which rely on the interaction of turbulence and rotation. The first involves the scale-by-scale transfer of energy known as the inverse cascade [2]. Large-scale vortices are known to be generated by this mechanism. The other mechanism relies on direct transfer of energy to the largest scales. It is hard to disentangle these two mechanisms in experiments and in simulations. Direct calculation of statistics and quasilinear direct numerical simulation (DNS) calculations have demonstrated that jets can be formed by the direct mechanism, not relying on an inverse cascade [3–6]; see also Ref. [7].

Nonequilibrium statistical mechanics can be used to understand universal aspects of fluids. Isotropic, homogeneous, turbulence is at present beyond the reach of a complete statistical theory. By contrast, inhomogeneous flows such as jets may be accessible to direct statistical simulation (DSS), that is, methods solving directly for the statistics of the flow [8]. DSS offers the possibility of a deeper understanding of fluid dynamics, as well as a practical speedup in obtaining statistics [5]. In the limit of small driving and dissipation, equilibrium statistical mechanics is a powerful tool for understanding quasi-two-dimensional flows (for a review, see Ref. [9]). Here and below the word "equilibrium" refers to the limiting case for which the rates of forcing and dissipation go to zero. Away from equilibrium, stochastic structural stability theory (SSST) [4,10,11] is one approach that has been explored to understand the formation and maintenance of jets. Here we instead investigate systematic expansion in equal time, but spatially nonlocal, cumulants of the flow. When truncated at second order, the cumulant expansion (denoted CE2) is closely related to SSST [6], but it is only the starting point for a perturbative expansion in higher cumulants.

This Letter examines the accuracy of DSS at CE2 as a representation of the statistics of turbulent flows driven away from equilibrium. CE2 includes the interaction of mean flows with eddies to drive eddies and that of eddies with eddies to drive mean flows, but removes the interaction of eddies with eddies in the evolution equation for the eddies [12], an interaction that has been termed the eddy-eddy nonlinearity by Srinivasan and Young [6]. Here eddies are formally the fluctuations about the zonal mean flow. It has been argued [9,13] that CE2 is an exact representation in the quasiequilibrium limit, but the domain of validity of such a truncation remains largely untested. We conduct numerical experiments to investigate the accuracy of DSS at CE2 for systems removed from quasiequilibrium by considering a model problem of the driving of jets by small-scale forcing on a β plane. This system has been studied extensively within the framework of DNS in both the fully nonlinear and quasilinear regimes [6,14]. Although this model is the simplest that includes all the requisite features for our purposes, i.e., anisotropy, nontrivial long-range correlations, and mean flows, we note that it is a rigorous test of statistical methods in that it is stochastically driven and translationally invariant in two directions, with only the emergence of jets spontaneously breaking the latitudinal symmetry [6] and leading to inhomogeneity. We return to this in the discussion at the end of the Letter.

The β plane we use is periodic in both x (longitude) and y (latitude), with the domain of size $2\pi \times 2\pi$. The motion of the incompressible fluid is damped by a single friction κ and by small-scale dissipation that absorbs structures at the finest scales. (Some models examined in Ref. [11] have friction damping the fluctuations 10 times greater than that slowing the zonal mean flow.) The fluid is driven by random (stochastic) forcing η. This type of stochastic forcing is widely used as a model of small-scale processes that inject energy into the fluid, with the small and fast scales acting as a random influence on the large and slow scales [15–17]. The time evolution of the relative vorticity $\zeta \equiv \hat{z} \cdot (\vec{\nabla} \times \vec{u})$ is given by, for example [18],

$$\partial_t \zeta + J(\psi, \zeta) + \beta \partial_x \psi = -\kappa \zeta + \nu \nabla^2 \zeta + \eta, \quad (1)$$

$$\zeta = \nabla^2 \psi, \quad (2)$$

where $J(a, b) = \partial_x a \partial_y b - \partial_y a \partial_x b$. Here ψ is the stream function and the fluid velocity $\vec{u} = (u, v) = (-\partial_y \psi, \partial_x \psi)$; we have set the deformation radius of the flow to be infinite. The forcing is random with a short (but nonzero) renewal time ($0.1 \leq r_t \leq 1$) and concentrated in the spectral band of wave numbers $k_{\min} \leq |k_x|, |k_y| \leq k_{\max}$ (for these runs $k_{\min} = 7$, $k_{\max} = 10$). The amplitude of the forcing is chosen from a Gaussian distribution with standard deviation a_η. This is a popular choice of forcing; a detailed discussion of the role of forcing in DNS of such problems is given in Ref. [14].

Rhines [19], who investigated the unforced system, demonstrated how correlations between nonlinear Rossby waves could lead to the generation of zonal flows and identified the scale at which zonal flows become important in mediating the dynamics of these waves (see, e.g., Ref. [20]). This "Rhines scale" is given by $L_R = (2U/\beta)^{1/2}$, where U is the rms velocity of the flow, and occurs when the second and third terms of Eq. (1) are comparable (and are comparable with the frictional term [21]). There has been much research into the importance of this length scale for the ultimate latitudinal scale of jets (see, e.g., Ref. [22] and the references therein), but it is also becoming clear that the dynamics of zonation is also controlled by another length scale L_ε [23], which measures the intensity of the forcing relative to the background potential vorticity gradient. For the simple β-plane model $L_\varepsilon = 0.5(\varepsilon/\beta^3)^{1/5}$, where ε is the energy input rate of the stochastic forcing η.

The ratio of these two length scales has been proposed, for models with small-scale forcing, to play a critical role in determining the strength and stability of jets [18,24], for cases where the same damping is applied to the mean flows and the turbulent fluctuations. This local measure, termed the zonostrophy index, is given by $R_\beta \equiv L_R/L_\varepsilon = U^{1/2}\beta^{1/10}/2^{1/2}\varepsilon^{1/5}$. In general, if the zonostrophy index is large, then strong stable jets are found, while for small R_β the jets are weaker, meander more, and no staircase is formed [14]. The zonostrophy index is therefore a measure of how far the system is driven out of equilibrium. Note that R_β can also be written (on balancing the energy input with the dissipation via friction $\varepsilon \sim \kappa U^2$) in terms of the ratio of an advective time on the Rhines scale to a dissipative time scale ($F_\beta = \kappa L_R/U$), i.e., $R_\beta = F_\beta^{-1/5}$. Hence the quasiequilibrium limit is given by $R_\beta \to \infty$. Recent estimates have put R_β between 5 and 6 for flows on the surface of Jupiter [25], while $R_\beta \sim 2$ for oceanic jets [18]. We note that the zonostrophy index might not be the only parameter controlling the dynamics of the jets. It has been shown that if the forcing length scale remains important, then the dynamics is controlled by two nondimensional parameters separately [6], and there is a regime given by a chain inequality for which R_β is the only important nondimensional parameter [18]. Nonetheless, even in this regime R_β does give *a* measure of lack of equilibrium.

DNS is performed using a pseudospectral scheme optimized for parallel machines [26]. For these simulations we utilize resolutions of up to 2048^2. The forcing is applied at moderate scale (with $r_t a_\eta^2 = 0.01$ for all calculations) and the system is evolved from rest until a statistically steady state is reached. Figure 1 shows a snapshot of the vorticity and zonally averaged vorticity for a state with 3 zonal jets. For this calculation $R_\beta = 2.12$ and the jet is well removed from the quasiequilibrium limit. We note that this limit is difficult to simulate in DNS, requiring long integrations. Nonetheless, the Hovmöller diagram in Fig. 2(a) of the (t, y) dependence of the mean flow together with a running time average calculated from the midpoint of the calculation shows that the zonal flows do not meander too much in space and well-defined averages can be obtained—though we note that lengthy integrations of the dynamics are required for meaningful flow statistics. Figure 2(b) shows

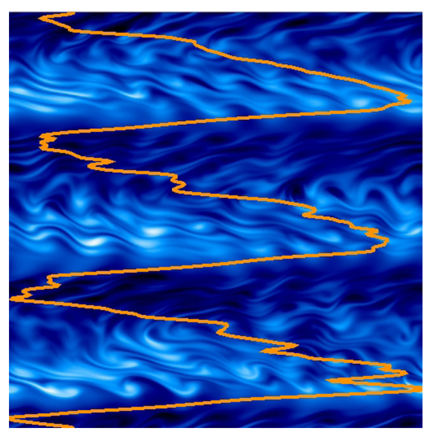

FIG. 1 (color online). Snapshot density plot of vorticity together with zonal mean vorticity profile of jets found by DNS. The parameters are $\kappa = 10^{-3}$, $\nu = 10^{-4}$, $\beta = 16$. For these parameters, $R_\beta = 2.12$.

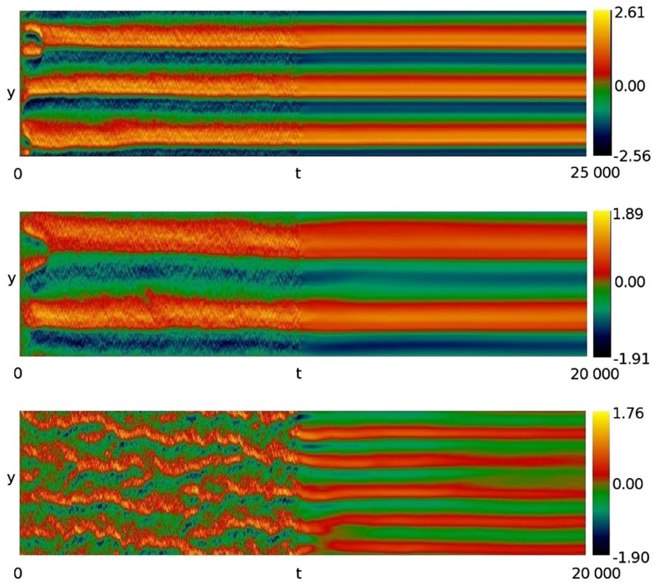

FIG. 2 (color online). Hovmöller diagrams of zonal mean relative vorticity versus time from DNS simulations. (a) Parameters as for Fig. 1 with $R_\beta = 2.12$. (b) Parameters as for (a) but $\beta = 8$ and $R_\beta = 1.98$; here L_R is increased by a factor $3/2$ from that in (a). (c) Parameters as for (a) but $\kappa = 10^{-2}$ and $R_\beta = 1.24$; here L_R is decreased by a factor $3/5$ from that in (a). A running time average commences at the midpoint of each diagram.

the corresponding diagram when R_β has been reduced to 1.98, which is achieved here by lowering β. For this case, even further from quasiequilibrium, the jets are still relatively steady, but the Rhines scale has changed sufficiently that now only two jets fit in the domain. This is consistent with the values of L_R given in the figure caption. For Fig. 2(c) R_β has been reduced to $R_\beta = 1.24$, achieved by increasing the friction. Here the jets meander significantly and merge, showing a large degree of spatiotemporal variation. In this respect they have the characteristics of observations and simulations of oceanic jets [27,28]. At different times either four or five jets appear, but on average there are five jets. Because of the temporal variability, the time average of the jet velocity is much smaller than the instantaneous jet speeds.

Expansion of equal-time cumulants at order CE2 is straightforward for Eq. (1). Let $\boldsymbol{r} = (x, y)$ and $\boldsymbol{r}' = (x', y')$ and adopt a Reynolds decomposition by setting $\zeta(\boldsymbol{r}) = \langle\zeta(\boldsymbol{r})\rangle + \zeta'(\boldsymbol{r})$, where the angle brackets imply either an ensemble average or an average over longitude (x). We define the first cumulants as $c_\zeta = \langle\zeta(\boldsymbol{r})\rangle = c_\zeta(y)$ and $c_\psi(\boldsymbol{r}) = \langle\psi(\boldsymbol{r})\rangle = c_\psi(y)$, where the relationship between these is given by $c_\zeta = \partial^2_{yy} c_\psi$. We may then define the second cumulants as follows: $c_{\zeta\zeta}(\boldsymbol{r}, \boldsymbol{r}') = \langle\zeta'(\boldsymbol{r})\zeta'(\boldsymbol{r}')\rangle$, and note that for this system $c_{\zeta\zeta}$ depends on the two local latitudes and the difference between the longitudes $\xi = x - x'$, i.e., $c_{\zeta\zeta}(\boldsymbol{r}, \boldsymbol{r}') = c_{\zeta\zeta}(y, y', \xi)$ [12]. Corresponding definitions arise for the derived second cumulants $c_{\psi\zeta}$ and $c_{\zeta\psi}$, i.e., $c_{\psi\zeta}(\boldsymbol{r}, \boldsymbol{r}') = \langle\psi'(\boldsymbol{r})\zeta'(\boldsymbol{r}')\rangle = c_{\psi\zeta}(y, y', \xi)$ and similarly for $c_{\zeta\psi}(y, y', \xi)$. With these definitions the equations for cumulant hierarchy, truncated at second order, are

$$\partial_t c_\zeta = -(\partial_y + \partial_{y'})\partial_\xi c_{\psi\zeta}\big|^{\xi\to 0}_{y\to y'} - \kappa c_\zeta + \nu \partial^2_{yy} c_\zeta, \quad (3)$$

$$\begin{aligned}\partial_t c_{\zeta\zeta} = {} & \partial_y c_\psi \partial_\xi c_{\zeta\zeta} - \partial_y[c_\zeta(y) - \beta y]\partial_\xi c_{\psi\zeta} \\ & - \partial_{y'} c_\psi \partial_\xi c_{\zeta\zeta} + \partial_{y'}[c_\zeta(y') - \beta y']\partial_\xi c_{\zeta\psi} \\ & + \nu(\nabla^2 + \nabla'^2)c_{\zeta\zeta} - 2\kappa c_{\zeta\zeta} + \Gamma. \end{aligned} \quad (4)$$

Here Γ is the covariance matrix of the stochastic forcing that enters into Eq. (4) as a deterministic source term localized at the same wave numbers as for the DNS [5] and with an amplitude a_Γ that is given $a_\Gamma = r_t a_\eta^2$. Equations (3) and (4) constitute a realizable closure, and in the absence of damping and forcing, conserve linear momentum, energy, and enstrophy. Equations (3) and (4) are integrated forward in time using a pseudospectral integrating factor or Adams-Bashforth numerical scheme. The integrations were performed at a typical resolution of 16×128. Restricting $|k_x| < 16$ does not amount to a further approximation beyond CE2, because, for this problem, at level CE2 only modes with zonal wave numbers less than those of the stochastic forcing are excited.

Recall that for R_β large the system is in quasiequilibrium, dominated by strong jets, and CE2 should provide an accurate representation of the statistics of the fluid flows. A typical evolution of the cumulant system is shown in Fig. 3(a). After some initial transients where jets are driven with a relatively small latitudinal extent, broader jets emerge via a series of jet mergings. Similar jet-merging behavior has been observed both in DNS and in the strong jet simulations of SSST [10], and also in the weakly nonlinear description of zonal jets [29]. The system eventually reaches a statistically steady state, represented by a simple fixed point of the cumulant equations.

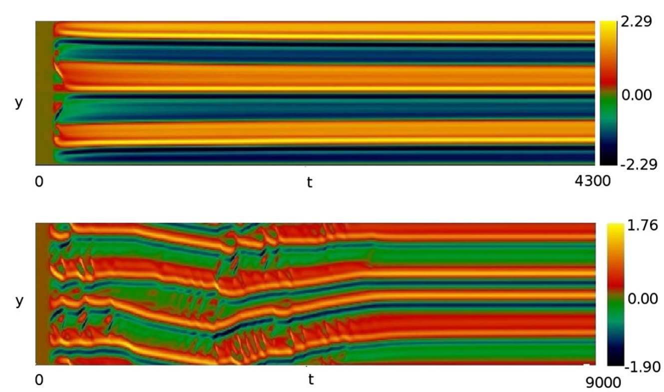

FIG. 3 (color online). Hovmöller diagrams of relative vorticity from CE2. (a) Parameters as for Fig. 2(a). (b) Parameters as for Fig. 2(c).

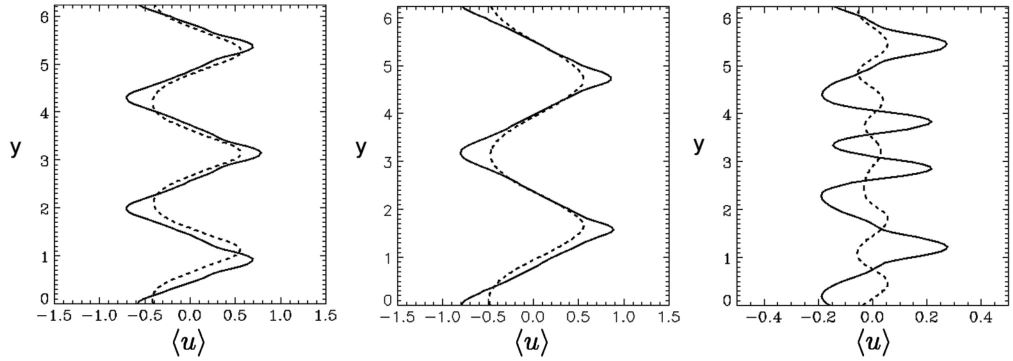

FIG. 4. Comparison of mean zonal velocity from DNS (dashed lines) and CE2 (solid lines) for parameters as in Figs. 2(a)–2(c) for which $R_\beta = 2.12$, 1.98, and 1.24.

The calculations were repeated at a range of R_β and compared with the (zonal and time averages of the) DNS solutions described earlier. Figure 4 shows comparisons of the zonal velocity in the jet from DNS averaged over both x and time with that achieved from DSS at CE2 for $R_\beta = 2.12$ and $R_\beta = 1.98$. The agreement in the first cumulant at these levels of disequilibrium is good; CE2 reproduces both the correct number of jets and their strength, although CE2 slightly overestimates the average jet strength—a characteristic in common with quasilinear DNS of jets [6]. However, close examination of the second cumulant reveals that CE2 struggles to reproduce the cross-correlation patterns (or teleconnections) from DNS for these parameters. The left-hand panel of Fig. 5 shows the second cumulant as accumulated from the DNS solution of Fig. 1. The figure shows the cross correlation of the vorticity statistics with respect to a test point. The second cumulant is localized in latitude, with some recurrent correlations occurring on the jet spacing, while the structure in longitude contributions both from wave number $k_x = 1$ and from the scale of the forcing. Examination of the spatiotemporal dynamics of the system indicates that the $k_x = 1$ contribution arises from a domain-scale meandering of the jet, termed "satellite modes" by Ref. [30]. The right-hand panel of Fig. 5 shows that CE2 reproduces

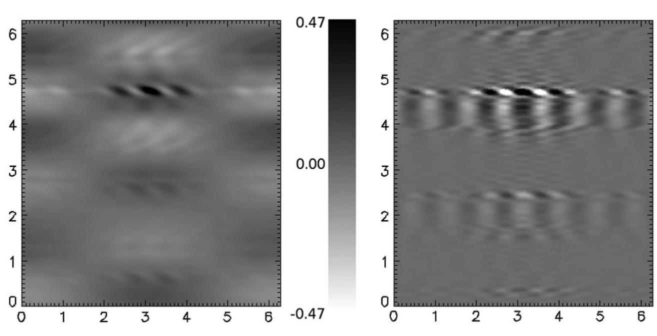

FIG. 5. Second cumulant $c_{\zeta\zeta}$ as calculated from DNS (left) with $R_\beta = 2.12$ and the corresponding CE2 solution (right). Cross correlation with respect to a test point located at $(\pi, 4.7)$.

the contributions to the second cumulant at the longitudinal scales of the forcing, but is incapable of reproducing the contribution from the satellite modes, when the system is this far from equilibrium. Interestingly these modes are also absent from quasilinear DNS calculations [6], which would seem to indicate that they arise as a result of eddy + eddy → eddy interactions.

For systems driven even further from equilibrium, CE2 struggles not only to reproduce all the structures of the second cumulant, but also the number of jets and their strength. As noted earlier, for smaller R_β the jets are more intermittent and meander more. Although zonal averages can be calculated, the constant meandering of the jets in latitude reduces the average jet strength. CE2 eventually settles down to a fixed point though we do not believe this to be a unique solution. The solution overestimates the strength of the jets and therefore the Rhines scale associated with them; hence, CE2 has a tendency to underestimate the number of jets as shown in Figs. 3(b) and 4(c).

This Letter has demonstrated that DSS as approximated by CE2 performs well in directly calculating the statistics for β-plane turbulence in quasiequilibrium. It confirms the earlier result [5,6] that zonal jets do not require an inverse cascade to be driven, but can arise as the result of Reynolds stresses alone. However, and importantly, we have shown that as the system is removed further from equilibrium by reducing the zonostrophy parameter R_β, CE2 can significantly overestimate jet strengths and predict the incorrect number of jets. We hypothesize that for such systems higher order cumulant expansions are required. If truncated at third order (CE3) the cumulant expansion includes eddy + eddy → eddy interactions and should perform better in predicting statistics for out-of-equilibrium systems. The potential utility of CE3 has been demonstrated for the problems of an isolated vortex [31] and fluid flow relaxing to a prescribed jet [32]. We conclude by noting that although we have stressed the limitations of CE2, we believe that the local β-plane system driven stochastically is one of the stiffest tests of this method; it is very difficult in both DNS and DSS to reach the quasiequilibrium limit, although progress may be achieved by utilizing DSS that implements semi-implicit time-stepping. Nevertheless, CE2 provides a good qualitative description of the first cumulant for systems where the important competing effects arise from inhomogeneity, anisotropy, and turbulent fluctuations about a nontrivial basic state.

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