Observation of Star-Shaped Surface Gravity Waves

Jean Rajchenbach,[1,*] Didier Clamond,[2] and Alphonse Leroux[1]

[1]*Laboratoire de Physique de la Matière Condensée (CNRS UMR 7336) Université de Nice—Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 2, France*

[2]*Laboratoire Jean-Alexandre Dieudonné (CNRS UMR 7351) Université de Nice—Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 2, France*

(Received 26 June 2012; published 28 February 2013; corrected 16 April 2013)

We report a new type of standing gravity wave of large amplitude, having alternatively the shape of a star and of a polygon. This wave is observed by means of a laboratory experiment by vertically vibrating a tank. The symmetry of the star (i.e., the number of branches) is independent of the container form and size, and can be changed according to the amplitude and frequency of the vibration. We show that a nonlinear resonant coupling between three gravity waves can be envisaged to trigger the observed symmetry breaking, although more complex interactions certainly take place in the final periodic state.

DOI: 10.1103/PhysRevLett.110.094502 PACS numbers: 47.35.Bb, 47.35.Fg, 47.54.−r

Nonlinear and dispersive effects in water waves give rise to remarkable phenomena, such as solitary and freak waves. These wave phenomena, originally observed at a liquid surface, turned out to have analogues in a number of other domains involving nonlinear waves. For example, solitary waves have also been recognized in optical fibers [1], and "freak" waves, which are giant waves of very short lifetime [2–4], have been identified in fiber optics [5] and in plasmas [6]. Another remarkable effect of nonlinearities is to give rise to patterning [7]. For example, "horseshoe" waves [8] have been shown to result from the nonlinear interactions between five waves [9,10]. Nevertheless, although the existence of a large variety of different waves is expected as a result of nonlinearities, experimental evidence of new types of waves are noticeably scarce. In this Letter, we report the observation of a new type of standing waves, displaying alternatively a starlike and a polygonal shape. These waves are observed at the free surface of a liquid submitted to vertical sinusoidal vibrations.

Experimental setup and observations.—The system studied is a fluid layer of about 1 cm deep; the liquid chosen for the investigations is a silicon oil, which, like water, displays a Newtonian rheological behavior. The kinematic viscosity is 10^{-5} m^2/s (i.e., 10 times that of water), and the surface tension is 0.02 N/m. Experiments are conducted with containers of various shapes (rectangular, circular) and of various sizes (from 7 to 20 cm in size or in diameter). The fluid vessel is mounted on a shaker and experiences a vertical sinusoidal motion, with a frequency $\Omega/2\pi$ ranging typically from 7 to 11 Hz. The amplitude of the cell oscillations can be driven up to 20 mm and the surface deformations are recorded by means of a fast camera (250 frames per second).

For the sake of clarity, we describe first the results obtained in a cylindrical container (9 cm diameter) vibrated with a frequency $\Omega/2\pi$ equal to 8 Hz, and with a filling level of 7 mm. For small oscillation amplitudes, we observe at the free surface of the liquid layer "meniscus ripples" originating from the contact line between the free surface and the inner wall of the container and propagating toward the center of the cell. These ripples oscillate with the same frequency as the driving, and the damping lengths are small compared to the radius of the container. Increasing the vibration amplitude up to 1.55 mm, we observe (see Fig. 1 and movie 1 in the Supplemental Material [11]) two contrapropagative, axisymmetric gravity waves, with a period T which is twice that of the forcing (i.e., $T = 4\pi/\Omega$) as it is expected for parametrically forced waves [12,13]. When the circular crest of the

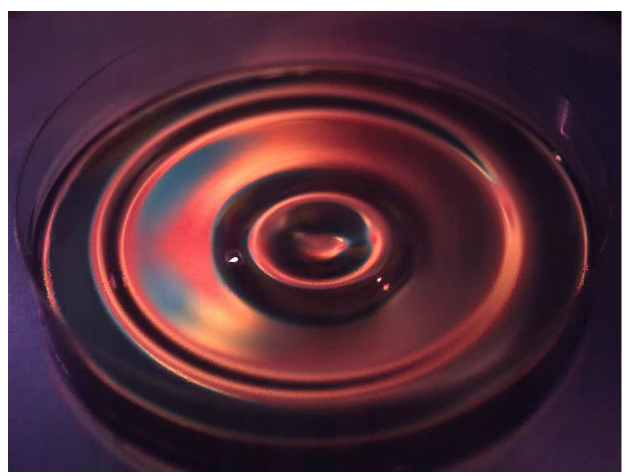

FIG. 1 (color online). Axisymmetric surface waves in a cylindrical container (diameter 9 cm, filling level 7 mm). These waves are parametrically excited by a vertical sinusoidal motion of the container (vibration amplitude = 1.70 mm) and oscillate subharmonically with the driving frequency (here $\Omega/2\pi = 8$ Hz). The inner and outer crests move contrapropagatively, and experience a phase shift when crossing (see movie 1 in the Supplemental Material [11]).

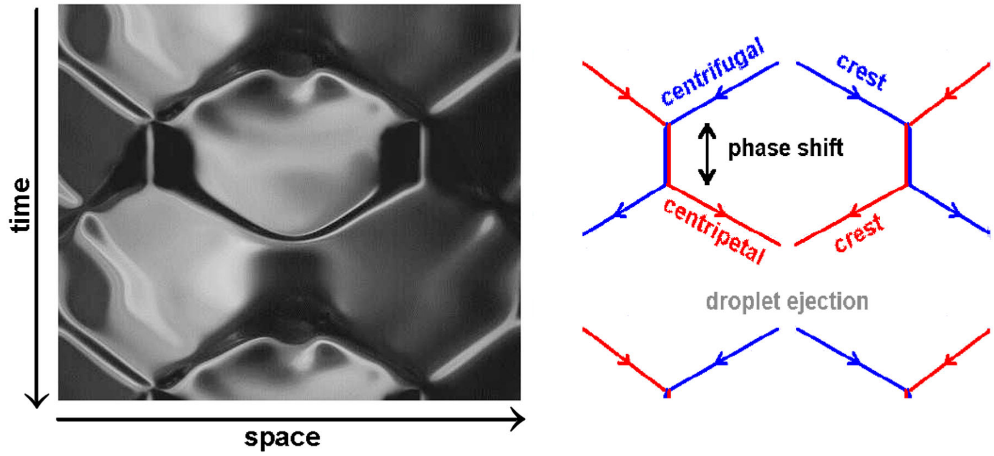

FIG. 2 (color online). Same experimental conditions as in Fig. 1. The present spatiotemporal diagram corresponds to the time evolution of the line of pixels passing through the center of the vessel. The line of pixels is plotted horizontally, and the time here is downward. This plot allows us to visualize the motion of the contrapropagative crests, and to measure the a phase delay when crossing. For these experimental conditions, the phase delay is equal to 0.05 s.

centripetal wave focuses to the center of the container, an upward jet is formed which breaks into a droplet. It is interesting to point out that, when the crests of the two centrifugal and centripetal axisymmetric waves are crossing, they do not simply superimpose, but they also experience a phase shift (see Fig. 2). More precisely, the crests remain in spatial coincidence during a typical time of 0.05 s for the above experimental parameters. The phase delay phenomenon during crossing has been recognized in the case of two crossing plane solitary waves, and testifies to a strong nonlinear coupling between the waves [14]. Still increasing the vibration amplitude to 1.85 mm, we notice the appearance of five corners in the crest line when the centrifugal and centripetal waves are crossing (see Fig. 3 and movie 3 in the Supplemental Material [11]). These tips signal the breaking of the rotational symmetry. Finally, for a typical vibration amplitude of 1.95 mm, we observe a drastic change in the wave geometry. The surface pattern displays alternatively a star and a pentagonal shape, separated by a time interval of $2\pi/\Omega$ [see Figs. 4(a) and 4(b) and movie 4 in the Supplemental Material [11]]. A remarkable feature is that these alternate star-polygon-shaped waves are independent of the container size and shape. Identical patterns are observed in larger circular or rectangular containers [Figs. 5(a) and 5(b)]. Note that we have also observed stars and polygons with other symmetries (third, fourth, and sixth order), merely by varying the frequency and the amplitude of vibration [see Fig. 6 and movies 6(a) and 6(b) in the Supplemental Material [11]]. Note also that the system exhibits hysteresis, meaning that for the same forcing parameters different patterns can be observed according to the forcing history. It is therefore not possible to establish a phase diagram related to the symmetry as a function of the forcing parameters.

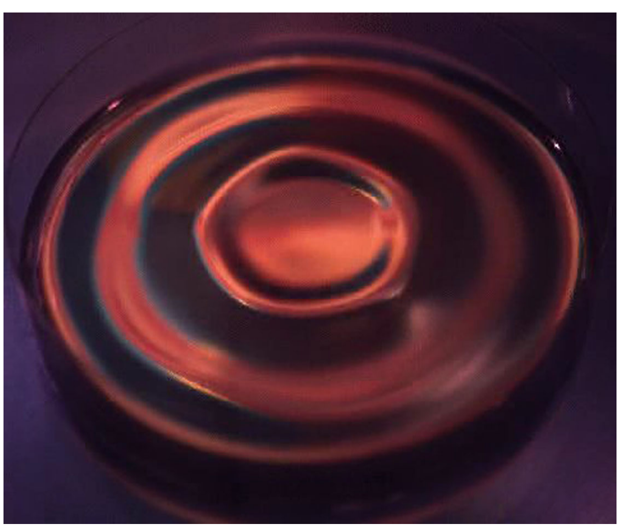

FIG. 3 (color online). For a larger vibration amplitude of the cell, we observe a deformation of the axisymmetric crest, with the appearance of five corners. This is the signature of a symmetry breaking (filling level 7 mm, $\Omega/2\pi = 8$ Hz, vibration amplitude 1.85 mm) (see movie 3 in the Supplemental Material [11]).

It must be emphasized that these waves are extreme: (i) the wave amplitude can be of the order of 2 times the liquid mean depth; (ii) in the trough, the depth is reduced to a film of less than 1 mm thick. Thus, these are highly nonlinear waves appearing in the context of shallow liquid (i.e., the wavelength/depth ratio, about 5–7, is large). In other words, we deal with large standing cnoidal waves.

Theoretical explanation.—Our interpretation is inspired by those of Mermin and Troian [15] and Pomeau and Newell [16] for quasicrystals, and that of Edwards and Fauve [17] for the formation of quasipatterns in capillary waves. It is noteworthy that in the present experiments we have $|\boldsymbol{k}| \ll 1/\ell_c$ (ℓ_c is the capillary length), so that here surface tension effects are negligible compared to gravity effects, and therefore we are dealing with pure gravity

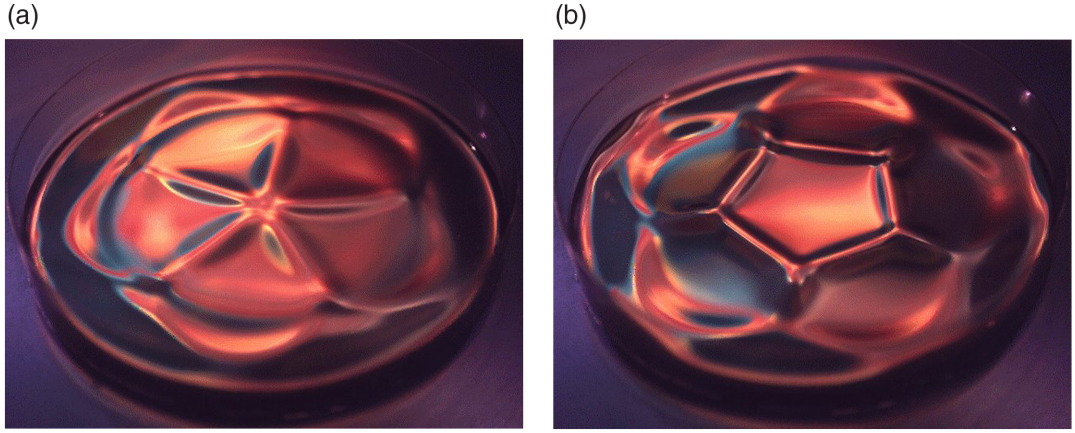

FIG. 4 (color online). A new type of standing wave appears for a vibration amplitude of 1.95 mm (filling level 7 mm, $\Omega/2\pi = 8$ Hz), having alternatively the shape of a five-branched star (a) and of a pentagon (b). The occurrence of these shapes is separated by an interval of time which corresponds to the forcing period, i.e., half the pattern period (see movie 4 in the Supplemental Material [11]).

waves. Our explanatory scheme involves a nonlinear resonant interaction between three surface waves. The three wave resonance conditions read as $\omega_1 \pm \omega_2 \pm \omega_3 = 0$ and $\boldsymbol{k}_1 \pm \boldsymbol{k}_2 \pm \boldsymbol{k}_3 = 0$ (ω_i and $\boldsymbol{k}_i$ are the angular frequencies and the wave vectors) [18,19]. These conditions can be simultaneously satisfied in capillary-gravity waves [20–24], but the three-wave resonance mechanism was considered up to now as irrelevant for the pure gravity waves that we are facing [18]. The reason is that the relation of dispersion of undamped, unforced gravity waves reads as $\omega \propto |\boldsymbol{k}|^\alpha$ with $\alpha \leq 1$ ($\alpha = 1/2$ in deep water, $\alpha = 1$ in shallow water), so that the above resonance conditions cannot hold. However, we show that this three-wave resonance mechanism is actually relevant to trigger the reported phenomenon, because the relation of dispersion is significantly modified by the dissipation and forcing. We will show explicitly below the dispersion relation taking into account dissipation and forcing, and then we will briefly explain how the amended relation of dispersion allows a three gravity wave resonant interaction and how the latter can select an m-fold symmetry.

It is well known [12,13,25–28] that the amplitude $\zeta(\boldsymbol{k}, t)$ of parametrically driven infinitesimal surface waves in finite depth, or Faraday waves [29], can be modeled by a damped Mathieu equation

$$\frac{\partial^2 \zeta}{\partial t^2} + 2\sigma \frac{\partial \zeta}{\partial t} + \omega_0^2[1 - F\cos(\Omega t)]\zeta = 0, \qquad (1)$$

where σ is the associated viscous attenuation, Ω is the forcing angular frequency, F corresponds to a dimensionless forcing (amplitude of the vertical acceleration divided by the gravity acceleration g), and $\omega_0 = \omega_0(|\boldsymbol{k}|)$ is the angular frequency of linear waves without damping and forcing [for linear water waves in finite depth h we have $\omega_0^2 = gk\tanh(kh)$ with $k = |\boldsymbol{k}|$ [19]]. The viscous attenuation term σ accounts for both the bulk dissipation

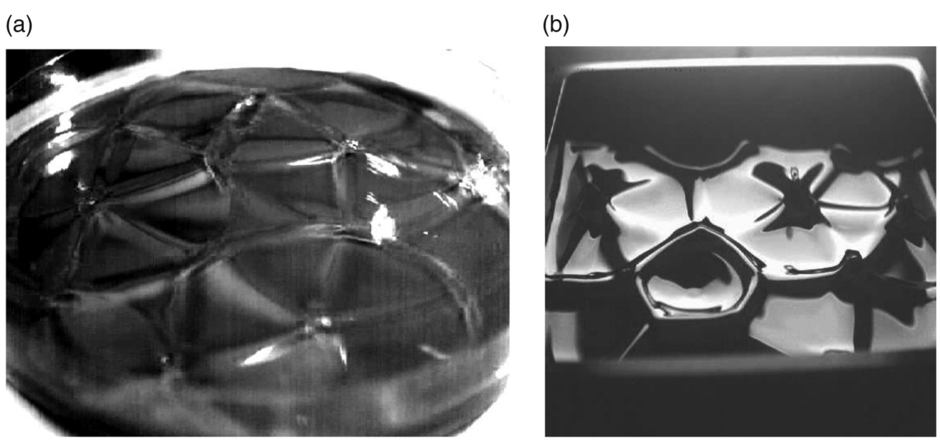

FIG. 5. For identical filling level, vibration parameters, and forcing history, the wave pattern is independent of the container shape and size. (a) In a cylindrical container of radius 17 cm, we observe a tiling of star-shaped waves. (b) In a square container (17 cm × 17 cm), we observe analogue patterns. Note that here adjacent pentagons and five-branched stars oscillate with a phase shift of π. This is an example of the possible solutions issued from the subharmonic instability.

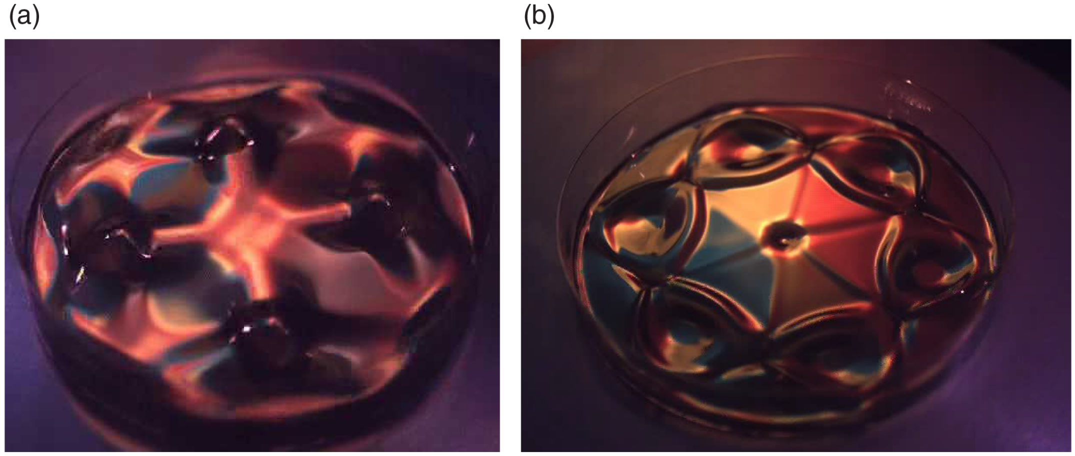

FIG. 6 (color online). Stars and polygonal waves with other symmetries can be observed with other vibration parameters or filling levels. (a) Symmetry of fourth order (filling level 8 mm, vibration amplitude 2.40 mm, $\Omega/2\pi = 12$ Hz). (b) Symmetry of 6th order (filling level 8 mm, vibration amplitude 2.90 mm, $\Omega/2\pi = 12$ Hz).

(proportional to νk^2 [30]) and the friction with bottom [proportional to $(\nu k^2)^{1/2}$ [31]]. It must be emphasized that Eq. (1) is linear, and is derived for infinitesimal waves in finite depth (i.e., *not* in shallow water). Here we deal with large amplitude, cnoidal waves, so that the validity of Eq. (1) is very limited. Nonetheless, Eq. (1) is valuable in providing insights on the mechanism triggering the formation of the patterns that we report here.

Systems obeying a damped Mathieu equation like Eq. (1) exhibit a series of resonance angular frequencies $n\Omega/2$ (the integer n is the *order* of the resonance) [32,33]. According to Floquet theory, bounded periodic solutions of Eq. (1) exist under some special relations between the parameters [34], these relations providing a dispersion relation (that cannot be expressed in term of elementary functions). Numerical investigations, using various expressions for σ, show that there are at most two wave number solutions of the dispersion relation for each n. This can be easily seen from the analytical expressions that we can derive in the limit of small F and small σ, that read as

$$\omega_0 \approx \frac{\Omega}{2}\left[1 \pm \sqrt{\frac{F^2}{16} - \frac{4\sigma^2}{\Omega^2}}\right], \quad (2)$$

for the subharmonic response, and

$$\omega_0 \approx \Omega\left[1 + \frac{F^2}{12} \pm \sqrt{\frac{F^4}{64} - \frac{\sigma^2}{\Omega^2}}\right], \quad (3)$$

for the fundamental one. Note that the damping introduces a threshold in the forcing amplitude giving rise to the formation of surface waves. In the limit of small F and σ, the thresholds are $F_1 \approx 8\sigma/\Omega$ for the subharmonic response ($n = 1$), $F_2 \approx \sqrt{8\sigma/\Omega}$ for the fundamental response ($n = 2$), and $F_3 \approx \sqrt[3]{8\sigma/\Omega}$ for the $3\Omega/2$ response ($n = 3$).

Unlike the case of undamped, unforced waves, relations (2) and (3) show that two modes with *different* wave numbers can oscillate at the *same* frequency. Therefore, according to the forcing amplitude, different cases must be distinguished.

(i) For $F < F_1$, there are no solutions of the dispersion relation (2). Physically, it means that there are no formations of parametric waves because the input of energy is not sufficient to overcome the viscous dissipation.

(ii) For $F_1 < F < F_2$, the excited modes are only those corresponding to subharmonic waves; i.e., they oscillate with angular frequency $\Omega/2$. If an infinite number of subharmonic waves with the same wave number (say k_1^-) are present, we observe an axisymmetric wave because, in a circular basin, the vertical wall boundary condition does not favor any particular direction.

(iii) For $F_2 < F < F_3$ (where F_3 is the threshold of the 3rd Mathieu's tongue), both subharmonic modes (oscillating at $\Omega/2$) and fundamental modes (oscillating at Ω) are excited. There are two wave numbers k_1^- and k_1^+ ($k_1^- \le k_1^+$) corresponding to the subharmonic mode, and two wave numbers k_2^- and k_2^+ ($k_2^- \le k_2^+$) for the harmonic one. All these modes interact nonlinearly. The simplest mechanism to be considered to explain the formation of waves with an m-fold rotational symmetry is the three-wave resonant coupling mechanism. Two subharmonic waves, of different wave vectors $\boldsymbol{k}_1^-$ and $\boldsymbol{k}_1^+$ and of identical angular frequencies $\omega_1 = \Omega/2$, interact between them and also interact with one fundamental mode, of wave vector $\boldsymbol{k}_2^-$ and of angular frequency $\omega_2 = \Omega$. Thus, the condition $\omega_1(\boldsymbol{k}_1^-) + \omega_1(\boldsymbol{k}_1^+) = \omega_2(\boldsymbol{k}_2^-)$ is automatically met. The additional condition to be fulfilled by wave vectors is $\boldsymbol{k}_1^- + \boldsymbol{k}_1^+ = \boldsymbol{k}_2^-$. This three-wave resonance condition naturally gives rise to the selection of a peculiar angle ($\boldsymbol{k}_1^-$, $\boldsymbol{k}_2^-$), which breaks the rotational invariance. Physically, the self-tuning of the angle between the wave vectors allows a continuous energy supply from two wave numbers to the third one. We have mentioned a three-wave resonant mechanism with wave numbers $\boldsymbol{k}_1^-$, $\boldsymbol{k}_1^+$, and $\boldsymbol{k}_2^-$, but another possible three-wave resonance involves $\boldsymbol{k}_2^+$ instead of $\boldsymbol{k}_2^-$. This multiplicity of possible three-wave resonances may be one cause of the observed hysteresis. Another cause is that, in viscous fluid, the parametric instability is subcritical, due to nonlinear effects, thus inducing a memory effect [35,36].

The m-branched stars and m-sided polygonal patterns correspond to the selection of an angle $\theta = 2\pi/m$, with m integer. Clearly, the above resonance criterion leads in general to m noninteger. In the latter case, the surface pattern appears unstationary, until a surface mode (not perfectly resonant) corresponding to m integer is locked. Once this mode (with m integer) is locked, it is seen to survive to moderate changes in the forcing parameters. This is a another possible origin for the observed hysteresis.

At this step, it is noteworthy that the above resonant coupling of three parametrically forced gravity waves can also be viewed somehow as a four-wave coupling, if we consider the forcing as the fourth wave.

Although the above model is capable to explain the triggering of a surface instability leading to the formation of m-fold symmetric gravity waves, it is insufficient to predict with accuracy the order of the final symmetries as a function of the forcing parameters. The reason is that Eqs. (2) and (3) are derived within the hypotheses of infinitesimal amplitude waves, while we are facing large amplitude cnoidal waves. Actually, the wave amplitudes intervene certainly in the dispersion relations. Moreover, considering sinusoidal waves as eigenmodes is a too crude approximation, unable to capture numerous physical properties [37]. The design of a highly nonlinear theory suited to large and steep cnoidal standing waves in shallow water remains a theoretical challenge for future studies.

We acknowledge Antoine Mangin (Société ACRI—Sophia-Antipolis, France) for his encouragements. This project has been partially supported by CNRS and Région PACA.

*Corresponding author.
Jean.Rajchenbach@unice.fr

[1] L. F. Mollenauer, R. H. Stolen, and J. P. Gordon, Phys. Rev. Lett. **45**, 1095 (1980).
[2] K. Dysthe, H. E. Krogstad, and P. Müller, Annu. Rev. Fluid Mech. **40**, 287 (2008).
[3] Special issue on Rogue Waves—Towards a Unifying Concept, edited by N. Akhmediev and E. Pelinovsky [Eur. Phys. J. Special Topics **185**, 1 (2010)].
[4] A. Chabchoub, N. P. Hoffmann, and H. Akhmediev, Phys. Rev. Lett. **106**, 204502 (2011).
[5] B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, N. Akhmediev, and J. M. Dudley, Nat. Phys. **6**, 790 (2010).
[6] W. M. Moslem, P. K. Shukla, and B. Eliasson, Europhys. Lett. **96**, 25002 (2011).
[7] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[8] M. Y. Su, J. Fluid Mech. **124**, 73 (1982).
[9] V. I. Shrira, S. I. Badulin, and C. A. Kharif, J. Fluid Mech. **318**, 375 (1996).
[10] D. Fructus, C. Kharif, M. Francius, Ø. Kristiansen, D. Clamond, and J. Grue, J. Fluid Mech. **537**, 155 (2005).
[11] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.110.094502 for movies.
[12] T. B. Benjamin and F. Ursell, Proc. R. Soc. A **225**, 505 (1954).
[13] J. Miles and D. Henderson, Annu. Rev. Fluid Mech. **22**, 143 (1990).
[14] M. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM Publications, Philadelphia, 1981); M. Ablowitz and D. Baldwin, Phys. Rev. E **86**, 036305 (2012).
[15] N. D. Mermin and S. M. Troian, Phys. Rev. Lett. **54**, 1524 (1985).
[16] Y. Pomeau and A. C. Newell, J. Phys. A **26**, L429 (1993).
[17] W. S. Edwards and S. Fauve, J. Fluid Mech. **278**, 123 (1994).
[18] O. M. Phillips, J. Fluid Mech. **9**, 193 (1960).
[19] J. Lighthill, *Waves in Fluids* (Cambridge University Press, Cambridge, England, 1978).
[20] S. Fauve, in *Dynamics of Nonlinear and Disordered Systems*, edited by G. Martinez-Meckler and T. H. Seligman (World Scientific, Singapore, 1995).
[21] H. W. Müller, Phys. Rev. E **49**, 1273 (1994).
[22] S. T. Milner, J. Fluid Mech. **225**, 81 (1991).
[23] W. Zhang and J. Viñals, J. Fluid Mech. **336**, 301 (1997).
[24] H. W. Müller *et al.*, in *Evolution of Spontaneous Structures in Dissipative Continuous Systems*, edited by F. Busse and S. Müller (Springer, Berlin, 1998).
[25] E. Meron, Phys. Rev. A **35**, 4892 (1987).
[26] S. Ciliberto and J. P. Gollub, J. Fluid Mech. **158**, 381 (1985).
[27] F. Simonelli and J. P. Gollub, J. Fluid Mech. **199**, 471 (1989).
[28] A. Kudrolli and J. P. Gollub, Physica (Amsterdam) **97D**, 133 (1996).
[29] M. Faraday, Philos. Trans. R. Soc. London **52**, 319 (1831).
[30] L. D. Landau and E. M. Lifschitz, *Fluid Mechanics* (Pergamon, Oxford, England, 1959).
[31] Nam Hong U, Bull. Russ. Acad. Sci. Phys. **57**, 131 (1993).
[32] M. Abramowitz and I. A. Segun, *Handbook of Mathematical Functions* (Dover, New York, 1970).
[33] L. D. Landau and E. M. Lifschitz, *Mechanics* (Pergamon, Oxford, England, 1960).
[34] K. Kumar and L. Tuckerman, J. Fluid Mech. **279**, 49 (1994).
[35] A. D. D. Craik and J. G. M. Armitage, Fluid Dyn. Res. **15**, 129 (1995).
[36] J. Rajchenbach, A. Leroux, and D. Clamond, Phys. Rev. Lett. **107**, 024502 (2011).
[37] J. V. Wehausen and E. V. Laitone, in *Encyclopaedia of Physics* (Springer-Verlag, Berlin, 1960), Vol. 9, p. 446.