

Thermodynamical Property of Entanglement Entropy for Excited States

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We argue that the entanglement entropy for a very small subsystem obeys a property which is analogous to the first law of thermodynamics when we excite the system. In relativistic setups, its effective temperature is proportional to the inverse of the subsystem size. This provides a universal relationship between the energy and the amount of quantum information. We derive the results using holography and confirm them in two-dimensional field theories. We will also comment on an example with negative specific heat and suggest a connection between the second law of thermodynamics and the strong subadditivity of entanglement entropy.

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In thermodynamics, when the total energy E of a system is increased, its entropy S grows accordingly. Its gradient is the definition of temperature T and this leads to the first law of thermodynamics $dE = TdS$. Since the entropy counts the number of microstates, this is a fundamental law which relates the amount of information included in a system to its total energy.

Thus, one may wonder if there is an analogous relation for general quantum systems which are far from equilibrium. One such example will be a system at zero temperature, i.e., a pure state. We can excite the system, for example, by producing massive particles. It is well known that a good measure of quantum information for a pure state is the entanglement entropy. Therefore, in this Letter we will study how the entanglement entropy for a certain region grows when we increase its energy. We will largely employ the AdS/CFT correspondence [1] and calculate the entanglement entropy holographically [2].

Consider an excited state in a d -dimensional conformal field theory (CFT). We assume it is almost static and translational invariant. The AdS/CFT correspondence argues that its ground state is equivalent to gravity on a $d + 1$ -dimensional anti-de Sitter space AdS_{d+1} [1]. The latter is called the gravity dual of the former. Thus, we start with the asymptotically AdS_{d+1} background:

$$ds^2 = \frac{R^2}{z^2} \left[-f(z)dt^2 + g(z)dz^2 + \sum_{i=1}^{d-1} (dx_i)^2 \right]. \quad (1)$$

Near the boundary $z \rightarrow 0$, we can assume $g(z) \simeq 1/f(z) \simeq 1 + mz^d$, where m is constant. We calculate the energy density T_{tt} of the excited state in the CFT from (1) by using the holographic energy stress tensor [3]:

$$T_{tt} = \frac{(d-1)R^{d-1}m}{16\pi G_N}. \quad (2)$$

We do not need to make any assumptions about the infrared region $z \rightarrow \infty$ for the argument below. For example, we

can have objects such as black branes or stars in the infrared region. The former has a horizon and is a thermal state, while the latter does not have any horizon and is dual to a zero temperature state.

To define the entanglement entropy S_A for a subsystem A , we divide the total system into A and B and consider the reduced density matrix on A , called ρ_A . ρ_A is defined by tracing out with respect to B : $\rho_A = \text{Tr}_B \rho_{\text{tot}}$, where ρ_{tot} is the density matrix for the total system. The entanglement entropy is defined by $S_A = -\text{Tr} \rho_A \log \rho_A$. In the gravity dual, we can calculate the holographic entanglement entropy by

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}, \quad (3)$$

where γ_A is a minimal area surface which ends at $z = 0$ on the boundary of A [2].

Our first choice of subsystem A is a strip defined by $0 < x_1 < l$, $-L/2 < x_{2,3,\dots,d-1} < L/2$, where L is taken to be infinite. We can parametrize the minimal surface γ_A by $x_1 = x(z)$. Then its area is computed as

$$\text{Area} = 2R^{d-1}L^{d-2} \int_{\epsilon}^{z_*} \frac{dz}{z^{d-1}} \sqrt{g(z) + x'(z)^2}. \quad (4)$$

By minimizing this area functional, we can determine the shape of $x(z)$. Finally, this leads to

$$\text{Area} = 2R^{d-1}L^{d-2} \int_{\epsilon}^{z_*} \frac{dz}{z^{d-1}} \sqrt{\frac{g(z)}{1 - \left(\frac{x}{z_*}\right)^{2(d-1)}}}, \quad (5)$$

$$l = 2 \int_0^{z_*} dz \frac{z^{d-1}}{z_*^{d-1}} \sqrt{\frac{g(z)}{1 - \frac{z^{2(d-1)}}{z_*^{2(d-1)}}}}, \quad (6)$$

where $z = z_*$ is the turning point of γ_A , i.e., the maximal value of z on γ_A .

Now, we impose an important assumption in this paper, that l is very small such that

$$ml^d \ll 1. \quad (7)$$

This means that γ_A is localized near the asymptotically AdS region and this is the reason why we can ignore the detail of infrared region. In this limit, we can expand (5) and (6) up to the first order perturbation by ml^d and eventually obtain

$$S_A = S_A^{(0)} + \Delta S_A, \quad (8)$$

$$\Delta S_A = \frac{R^{d-1}mL^{d-2}l^2}{32(d+1)G_N} \frac{\Gamma\left(\frac{1}{2(d-1)}\right)^2 \Gamma\left(\frac{1}{d-1}\right)}{\sqrt{\pi} \Gamma\left(\frac{1}{2} + \frac{1}{d-1}\right) \Gamma\left(\frac{d}{2(d-1)}\right)^2}.$$

$S_A^{(0)}$ is the holographic entanglement entropy in the pure AdS $_{d+1}$ calculated in Ref. [2]. Thus, ΔS_A measures how much S_A is increased in the excited state compared with the ground state of the CFT.

On the other hand, the increased amount of energy in the subsystem A is given by

$$\Delta E_A = \int dx^{d-1} \Delta T_{tt} = \frac{(d-1)mL^{d-2}R^{d-1}}{16\pi G_N}. \quad (9)$$

Therefore, we find the following relation:

$$\frac{\Delta S_A}{\Delta E_A} = \frac{\sqrt{\pi} \Gamma\left(\frac{1}{2(d-1)}\right)^2 \Gamma\left(\frac{1}{d-1}\right)}{2(d^2-1) \Gamma\left(\frac{1}{2} + \frac{1}{d-1}\right) \Gamma\left(\frac{d}{2(d-1)}\right)^2} l. \quad (10)$$

For another example, consider the case where A is given by a round ball with radius l : $\sum_{i=1}^{d-1} x_i^2 \leq l^2$. Its minimal surface γ_A is specified by $r = r(z)$. The area is computed as

$$\text{Area} = R^{d-1} \Omega_{d-2} \int_{\epsilon}^u \frac{dz}{z^{d-1}} r(z)^{d-2} \sqrt{g(z) + r'(z)^2},$$

where $\Omega_{d-2} = 2 \frac{\pi^{d-1/2}}{\Gamma(\frac{d-1}{2})}$ is the volume of S^{d-2} with the unit radius.

By solving the equation of motion for $r(z)$ derived from the above area functional, we find the following solution up to the first order of ml^d :

$$r(z) = \sqrt{u^2 - z^2} + m \frac{2u^{d+2} - z^d(u^2 + z^2)}{2(d+1)\sqrt{u^2 - z^2}}, \quad (11)$$

where we assumed regularity at $z = u$. The parameter u is a free positive constant and is related to the radius l of the subsystem A by $l = r(0) = u + mu^{d+1}/(d+1)$. In the end, we find the increased amount of S_A

$$\Delta S_A = \frac{\pi^{d-1/2}}{4(d+1)\Gamma\left(\frac{d-1}{2}\right)} \frac{R^{d-1}}{G_N} ml^d. \quad (12)$$

By computing ΔE_A from (2), we finally obtain

$$\frac{\Delta S_A}{\Delta E_A} = \frac{2\pi}{d+1} l. \quad (13)$$

If we take the limit (7), the entanglement entropy in CFTs, as shown in (10) and (13), satisfies a universal relation analogous to the first law of thermodynamics

$$T_{\text{ent}} \Delta S_A = \Delta E_A, \quad (14)$$

where the effective temperature (“entanglement temperature”) T_{ent} is proportional to the inverse of l :

$$T_{\text{ent}} = cl^{-1}. \quad (15)$$

c is an order one constant. When the subsystem A is a round sphere, we find $c = \frac{d+1}{2\pi}$.

In the field theoretic language, this argues that in strongly coupled large N gauge theories, the relations (14) and (15) are satisfied if we take the subsystem size l to be very small such that

$$T_{tt} l^d \ll R^{d-1}/G_N \sim O(N^2). \quad (16)$$

What we learn from (14) is the universal statement that the amount of information included in a small subsystem A is proportional to the energy included in A . The AdS/CFT predicts that the constant c in (15) is universal when we fix the shape of the subsystem A . Note that the source of the excitation energy is arbitrary. It can be a temperature increase or can be creations of massive objects at zero temperature. The condition (16) is crucial here. If we do not assume this, then T_{ent} defined by (14) is no longer universal and depends on the details of the excitations. In the gravity side, the result depends on the details of the infrared region. For example, in the AdS Black hole at temperature T , T_{ent} defined by (14) approaches (15) in the limit $l \rightarrow 0$, while T_{ent} becomes T in the opposite limit $l \rightarrow \infty$ as plotted in Fig. 1.

Finally, we would like to mention that there are earlier works [4,5], where the entanglement entropy for ground states $S_A^{(0)}$ was interpreted as thermal entropy. Also, the crossover between the entanglement entropy and thermal entropy has been discussed in Ref. [6].

Since we analyzed the gravity duals so far, it is useful to turn to a direct calculation in quantum field theories

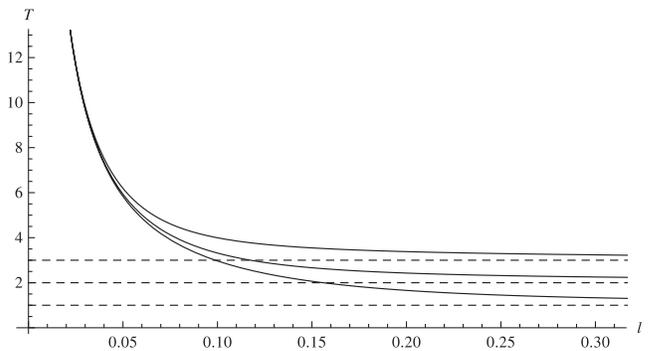


FIG. 1. The effective temperature $T_{\text{ent}} = \frac{d(\Delta E_A)}{d(\Delta S_A)}|_{l=\text{fixed}}$ as a function of l based on the holographic calculation using the AdS $_4$ black hole at temperature T . We set $T = 1, 2,$ and 3 .

(QFTs). However, at present, this has only been done for two-dimensional CFTs. First, we know the general formula of S_A [7] at finite temperature $T = \beta^{-1}$ when A is an interval of width l :

$$S_A = \frac{c}{3} \log\left(\frac{\beta}{\pi\epsilon} \sinh\frac{\pi l}{\beta}\right). \quad (17)$$

By expanding this in the limit $l \ll \beta$, we find

$$\Delta S_A = \frac{c\pi^2 T^2 l^2}{18} = \frac{Rm l^2}{48G_N} = \frac{\pi}{3} l \Delta E_A, \quad (18)$$

where we employed the standard relations $c = \frac{3R}{2G_N}$ [8] and $m = (2\pi T)^2$ in $\text{AdS}_3/\text{CFT}_2$. This agrees with our holographic calculation in (8). Moreover, if we set the width of interval as $2l$, then it also agrees with (13).

Also, ΔS_A was recently calculated for low-energy excitations on a cylinder with the circumference L_{cy} [9]:

$$\Delta S_A = \frac{2\pi^2}{3} (h + \bar{h}) \frac{l^2}{L_{cy}^2} = \frac{\pi}{3} l \Delta E_A, \quad (19)$$

where (h, \bar{h}) are the chiral and antichiral conformal weights of a primary operator.

In higher dimensional CFTs, we were not aware of relevant QFT calculations at present. Instead, our result (10) presents predictions: $\Delta S_A \propto l^2 L^{d-2} T^d$ at temperature T with $lT \ll 1$; $\Delta S_A \propto \Delta E_A l^d$ for a CFT_d on $R \times S^d$ (unit radius) with the subsystem radius $l \ll 1$, where ΔE_A is the conformal weight of the excited state.

We can extend our analysis to gravity duals of more general nonrelativistic critical points called the hyperscaling violating geometry [10]:

$$ds^2 = \frac{R^2}{r^2} (-r^{-(2(d-1)(z-1)/d-1-\theta)} dt^2 + r^{2\theta/d-1-\theta} dr^2 + dx_i^2), \quad (20)$$

where θ is the hyperscaling violating exponent and z is the dynamical exponent. We can realize this geometry as a solution in Einstein-Maxwell-scalar theory [10].

We heat up this system at temperature T and consider the holographic entanglement entropy for the strip subsystem with the width l . Strictly speaking, we need to embed everything into an asymptotic AdS space, where the IR geometry looks like (20). We define z_1 to be the scale where the hyperscaling violating geometry starts to appear. We are interested in the region $z_1 \ll l \ll T^{-1}$. In this case, the analysis of the holographic entanglement entropy is not affected by the presence of the asymptotically AdS_4 region and we can focus on the metric (20).

The finite temperature solution is the black brane solution given by multiplying $f(r)$ and $1/f(r)$ in front of dt^2 and dr^2 in (20), where $f(r) = 1 - (r/r_H)^{(d-1)(1+z/d-1-\theta)}$ [10]. The horizon is situated at $r = r_H$. We immediately find the thermal entropy as the horizon area, and then by using the first law of thermodynamics, we can calculate the energy

density. We can calculate ΔS_A as in the previous analysis (8). The final results look like $\Delta S_A/\Delta E_A = l^z/c_h$, where c_h is an order one constant. Thus, we find $T_{\text{ent}} \propto l^{-z}$, which is natural from the definition of the dynamical exponent z .

We would like to discuss an entanglement entropy counterpart of the positivity of specific heat, which is required by the second law of thermodynamics. A good example of this kind is the $D3$ -brane shell [11]. The near horizon geometry of $D3$ -brane shell is given by the metric:

$$ds^2 = \frac{R^2}{z^2 h(z)} \left(\sum_{\mu=0}^3 dx_\mu dx^\mu \right) + R^2 h(z) \left(\frac{dz^2}{z^2} + d\Omega_5^2 \right),$$

$$h(z) = 1 (z \leq z_0), \quad h(z) = z_0^2/z^2 (z \geq z_0). \quad (21)$$

This solution represents $D3$ -branes distributed at $z = z_0$ like a spherically symmetric shell. The geometry for $z > z_0$ is a flat spacetime $R^{1,9}$, while that for $z < z_0$ is the $\text{AdS}_5 \times S^5$. This is dual to a $N = 4$ super Yang-Mills in the Coulomb branch [11].

By exciting this system, we can make a small black hole at $z = \infty$. This is a Schwarzschild black 3-brane in flat spacetime and thus it has the negative specific heat. This solution is thermodynamically unstable and will finally decay into the standard AdS black hole solution.

Consider the holographic entanglement entropy for the strip with the width l in the $D3$ -brane shell. It is computed by finding the eight-dimensional minimal area surface $x = x(z)$ at a time $t = 0$. The area functional is given by

$$\text{Area} = 2\pi^3 R^8 L^2 \int_0^{z_*} \frac{dz}{z^3} h(z) \sqrt{x'(z)^2 + h(z)^2}, \quad (22)$$

where $z = z_*$ is the turning point of a connected surface. There is another candidate of the minimal surface which consists of two disconnected surfaces defined by $x = l/2$ and $x = -l/2$. We plotted the regularized area, which is defined by subtracting the area law divergence $\frac{L^2}{\epsilon^2}$ [12] from the area, in Fig. 2. The result looks similar to the one in the gravity duals of confining gauge theories [13]. Notice that there are two branches in the connected surface for a fixed value of l . The one with the lower area is sensible as it satisfies the strong subadditivity equivalent to the concavity $\frac{d^2 S_A}{dl^2} \leq 0$ [14–16]. However, the other one is not. Since we need the smallest area when there are several candidates of γ_A , we choose the lowest area surface. For a certain value of $l = l_c$, the disconnected surface is favored as it has a smaller area, where S_A shows an analogue of the phase transition as a function of l .

Next let us turn to the unstable surface which is not concave. Even though this surface does not contribute to S_A , it is intriguing to understand the meaning of its presence. This surface extends deep into the IR flat space region $z \rightarrow \infty$. Thus, it is natural to expect that this is related to a Schwarzschild black 3-brane in $R^{1,9}$. This makes us suspect that the negative specific heat is related to the violation of strong subadditivity. Remember that the entropy S and

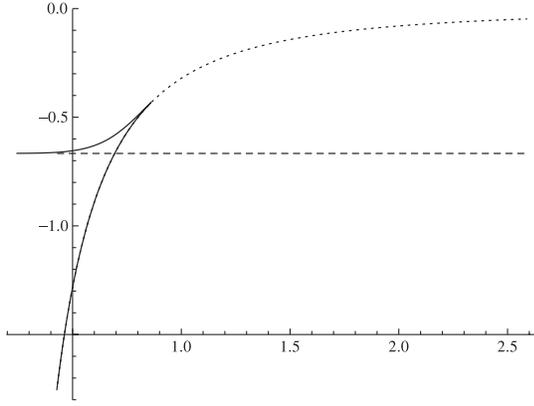


FIG. 2. The plot describes the regularize minimal surface area (divided by $\pi^3 R^8 L^2$) as a function of the width l in the $D3$ -brane shell. We choose $z_0 = 1$. The two thick curves correspond to the connected minimal surfaces, where upper one is not concave and thus is not physical. The horizontal dashed line represents the disconnected surface in the $D3$ -brane shell. The dotted curve is the result for the pure AdS.

energy E of a black p -brane in $R^{1,9}$ behave like $S_p \propto V_p T^{p-8}$ and $E_p \propto V_p T^{p-7}$, where V_p is the volume of the p -brane and T is its temperature. On the other hand, in the small l limit, we can show that the finite part of holographic entanglement entropy behaves as $[S_A]_{\text{finite}} \propto \frac{R^8 L^2 l^6}{z_0^8}$. By identifying V_p with the volume of subsystem A and the temperature T with $1/l$, we find that $[S_A]_{\text{finite}}$ agrees with the entropy of black 3-brane S_3 up to a numerical factor.

Finally, let us study the connection between the sign of the specific heat and the strong subadditivity. We assume that the finite part of the entanglement entropy in any dimension behaves like $[S_A]_{\text{finite}} = \alpha l^{q+1}$ for the strip subsystem A with the width l . Since we expect S_A is a monotonically increasing function of l , α is positive (or negative) when $q+1 > 0$ (or $q+1 < 0$). The strong subadditivity (i.e., concavity) requires $q \leq 0$. By identifying the temperature and volume as $T \propto 1/l^z$ and $V_p \propto l L^{p-1}$, we can relate S_A to the thermal entropy $S_{th} \propto V_p T^{-q/z}$. The positive specific heat requires $q \leq 0$. In this way, we find that the strong subadditivity is equivalent to the positivity of specific heat in this setup.

In conclusion, the main result of this paper is that the entanglement entropy in CFTs satisfies the first lawlike relation (14) with the universal entanglement temperature (15) when the subsystem A is so small that (16) is satisfied. This means that the variation ΔS_A is given by physical observables. We derived this from the AdS/CFT and confirmed this in 2D CFTs. An interesting future problem is to check this directly in higher dimensional QFTs.

There are many different ways to add the energy ΔE_A to the subsystem A . Consider a zero temperature setup where ΔE_A depends on an external parameter x such as the

distance between two interacting particles. Then we can express its force F_x in terms of ΔS_A as follows:

$$F_x = -\frac{d}{dx} \Delta E_A(x) = -T_{\text{ent}} \frac{d}{dx} \Delta S_A(x). \quad (23)$$

Though this might look like an entropic force [17,18], the sign is opposite. This force F_x acts so that it tries to reduce the entanglement entropy. In the gravity dual, F_x can be a gravitational force or some other forces which exist in the gravity theory. It is an interesting future problem to study the implication of (23).

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