Comment on "Trouble with the Lorentz Law of Force: Incompatibility with Special Relativity and Momentum Conservation"

Mansuripur argued in [1] that the Lorentz force law is incompatible with the special relativity theory and momentum conservation, based on an example in which a magnetic dipole does not suffer any torque from an electric charge in the system rest frame, but in other frames such a torque appears if the Lorentz law is used to compute the forces on it. However, here we show that there is no paradox if the "hidden momentum" of a magnetic dipole in the presence of an external electric field is taken into account.

The electric charge from Mansuripur's example [1] produces an electric field $\mathbf{E} = E\hat{\mathbf{z}}$ on the (infinitely small) magnetic dipole with dipole moment $\mathbf{m}_0 = m_0\hat{\mathbf{x}}$. Different models that use current loops to represent a magnetic dipole predict that the dipole acquires a hidden momentum $\mathbf{P} = \varepsilon_0\mathbf{m}_0 \times \mathbf{E}$ in the presence of the external electric field due to relativistic effects on the moving charges of the loops [2,3]. This hidden momentum is counterbalanced by the electromagnetic momentum obtained from the integral of $\varepsilon_0\mathbf{E} \times \mathbf{B} = \varepsilon_0\mathbf{E} \times (\mu_0\mathbf{H} + \mathbf{M})$ in the whole space. Since we have $\mathbf{H} = 0$ and $\mathbf{M} = \mathbf{m}_0\delta^3(\mathbf{r} - \mathbf{r}_0)$ in Mansuripur's example [1], $\mathbf{r}_0$ being the dipole position, the system total momentum is zero.

Now let us consider the same system in a reference frame that moves with velocity $\mathbf{V} = -V\hat{\mathbf{z}}$. The electric field in the position of the magnetic dipole is the same as before and there is no magnetic field. However, the magnetic dipole acquires an electric dipole moment $\varepsilon_0 m_0 V\hat{\mathbf{y}}$, such that there is a net torque $\mathbf{T} = E\varepsilon_0 m_0 V\hat{\mathbf{x}}$ acting on it [1]. But if we take the hidden momentum into account, there is no inconsistency. In the new frame this momentum is the same, $\mathbf{P} = -\varepsilon_0 m_0 E\hat{\mathbf{y}}$. If the magnetic dipole is at the origin of the system of coordinates at the origin of time, its position in the new frame is $\mathbf{r} = Vt\hat{\mathbf{z}}$ and the angular momentum is $\mathbf{L} = \mathbf{r} \times \mathbf{P} = \varepsilon_0 m_0 EVt\hat{\mathbf{x}}$, such that $d\mathbf{L}/dt = \mathbf{T}$. The torque is equal to the rate of change of the angular momentum caused by the movement of an object with hidden momentum, such that there is no angular acceleration of the dipole and no paradox arises with the use of the Lorentz law. There is a "hidden angular momentum" that increases in time but is counterbalanced by the electromagnetic angular momentum such that the system total angular momentum is constant in time.

The magnetic dipole moment of quantum systems like atoms and electrons cannot be described by classical current loops. So it is not possible to say if such objects have or do not have hidden momentum in the presence of an electric field based on the classical arguments from Refs. [2,3].

This issue is related to the Abraham-Minkowski debate about the momentum of electromagnetic waves in material media [4]. The eventual conclusion of the debate is that there are many different ways for dividing the total energy-momentum tensor of the system into electromagnetic and material parts, corresponding to different expressions for the electromagnetic momentum density, force, energy flux, etc., that lead to the same experimental predictions [2,4]. In previous works [5,6] we have shown that if the momentum and energy transferred to matter by an electromagnetic wave are computed by the use of the Lorentz law of force, we must consider the expression $\varepsilon_0\mathbf{E} \times \mathbf{B}$ for the electromagnetic momentum density, $\mathbf{E} \times \mathbf{B}/\mu_0$ for the electromagnetic energy flux and take the hidden momentum and a "hidden energy flux" into account to have energy and momentum conservation in different situations, as well as an agreement with Balazs's gedanken experiment [7]. Here we are confirming this fact. On the other hand, with the use of the Einstein-Laub force one must consider the expressions $\mathbf{E} \times \mathbf{H}/c^2$ for the electromagnetic momentum density, $\mathbf{E} \times \mathbf{H}$ for the electromagnetic energy flux, and disregard the hidden momentum. Since both formulations give the same experimental predictions when properly used, they are equally valid.

It is important to stress that the Lorentz force law can be written in a covariant way, such that it is automatically compatible with special relativity. Mansuripur's apparent paradox can be solved based on this fact, as discussed in Refs. [8–10]. It is also worth mentioning that similar apparent paradoxes regarding the torque on magnetic dipoles were discussed and solved many years ago [3,11,12].

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Note added.—After the submission of the first version of this work for publication, several related works were posted on the Internet based on essentially the same arguments that we present here. See [8–10,13], to quote a few.

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