Comment on "Trouble with the Lorentz Law of Force: Incompatibility with Special Relativity and Momentum Conservation"

In Ref. [1], a paradox involving a magnet in the presence of an electric field is presented: in the magnet's rest frame it seems oblivious to the electric field while according to an observer who sees the magnet moving it is now subject to a torque exerted by the same electric field. Therefore, while the magnet stands still in its rest frame, it seems to be compelled to rotate as seen by another inertial frame. According to Ref. [1], the Lorentz force is the one to blame for this paradox, being incompatible with special relativity. Here, we show that this alleged incompatibility is simply impossible. As is well known, the Lorentz force can be put in a covariant form. Anyone familiar with the geometrical formulation of special relativity should know that a specially covariant law *cannot* lead to incompatible descriptions of the same phenomenon in different inertial frames simply because the whole phenomenon can be seen as taking place on the four-dimensional Minkowski spacetime; no need to adopt *any* inertial frame. Different observers, who perceive the spacetime split differently into space and time, give different descriptions of this *one* phenomenon. But since all these descriptions are connected by the same four-dimensional view, they cannot be inconsistent with each other. The proposed covariant law accounts satisfactorily for the phenomenon in *all* inertial frames or in *none*.

Let us revisit the "charge-magnet paradox" but now considering the full four-dimensional picture enforced by special relativity. The neutrality of the magnet in its rest frame is encoded in its (bound) four-current density j^μ being purely spatial in this frame: $j^\mu u_\mu = 0$ (u^μ is the magnet's four-velocity). A purely electric field E^μ in this frame ($E^\mu = F^{\mu\nu}u_\nu/c$) is encoded in the electromagnetic tensor $F^{\mu\nu} = (u^\mu E^\nu - u^\nu E^\mu)/c$. Thus, the Lorentz four-force density, $f^\mu = F^{\mu\nu}j_\nu/c$, evaluates to $f^\mu = (j_\alpha E^\alpha)u^\mu/c^2$. Therefore, in the magnet's rest frame the Lorentz force has no spatial component: the Lorentz (three-)force is zero (see Fig. 1). However, jumping to the conclusion that the magnet is really oblivious to the electric field in its rest frame is incorrect and stands at the root of the "paradox." In fact, special relativity teaches us that $\partial_\mu T^{\mu\nu} = f^\nu$, where $T_{\mu\nu}$ is the energy-momentum tensor of the magnet (only the electric field is not included in $T_{\mu\nu}$). Projecting this equation in the u^μ direction, $\partial_\mu \pi^\mu = (j_\alpha E^\alpha)/c^2$ ($\pi^\mu = -T^{\mu\nu}u_\nu/c^2$ is the four-momentum density of the magnet in its rest frame), then using that the magnet's energy-momentum distribution is stationary in its rest frame ($\partial_\mu \pi^\mu = \mathrm{div}\vec{\pi}$), and finally integrating the spatial part $\vec{\pi}$ of π^μ give the total momentum of the magnet in its rest frame: $\vec{P} = \int d^3x \vec{M} \times \vec{E}/c^2$, where $\vec{M}$ is the magnet's magnetization. Therefore, one can easily anticipate that if the magnet moves with velocity $\vec{V}$ along the electric field direction, its total angular momentum will be time dependent. This *demands* a net torque $\vec{\tau} = \vec{V} \times \vec{P}$ which is exactly the one provided by the Lorentz force in the "moving frame" (see Fig. 1) [2]. The Lorentz force which in the "moving frame" is responsible for the torque $\vec{\tau}$ is the same which in the rest frame of the magnet induces the momentum $\vec{P}$. One cannot dismiss the latter while considering the former, as Ref. [1] does. No paradox here [3]. It is a quite *generic* feature of relativistic systems subject to external forces that the *total* momentum is *not* fully encoded in the motion of the center of mass-energy. The sole motivation of Ref. [1] of avoiding this kind of "hidden" momentum [7–11] is, therefore, misguided.

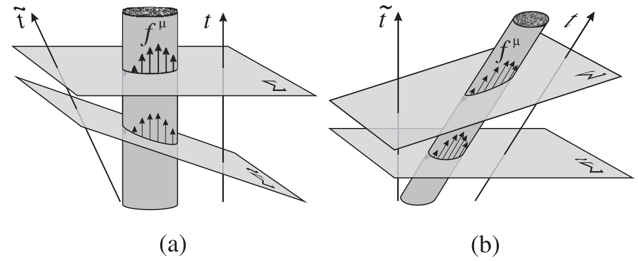

FIG. 1. Equivalent four-dimensional representations of a cross section of a uniformly magnetized sphere subject to a uniform electric field. (a) Privileging the rest frame of the sphere, where t is its proper-time coordinate and Σ its spatial three-surface. (b) Privileging a "moving frame," where $\tilde{t}$ is its proper-time coordinate and $\tilde{\Sigma}$ its spatial three-surface. The Lorentz four-force density is future directed where the electric field favors the current density (depicted in the figure) and past directed otherwise (on the opposite side). Note that it has null projection on Σ while having circulation on $\tilde{\Sigma}$.

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