Lifshitz Gravity for Lifshitz Holography

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(Received 21 November 2012; published 19 February 2013)

We argue that Hořava-Lifshitz (HL) gravity provides the minimal holographic dual for Lifshitz-type field theories with anisotropic scaling and a dynamical exponent z. First we show that Lifshitz spacetimes are vacuum solutions of HL gravity, without need for additional matter. Then we perform holographic renormalization of HL gravity, and show how it reproduces the full structure of the $z = 2$ anisotropic Weyl anomaly in dual field theories in $2 + 1$ dimensions, while its minimal relativistic gravity counterpart yields only one of two independent central charges in the anomaly.

DOI: 10.1103/PhysRevLett.110.081602 PACS numbers: 11.25.Tq, 04.50.−h, 04.60.−m

The concepts of scaling and the renormalization group have played a central role in organizing our understanding of quantum field theory (QFT) and statistical systems for half a century. Here we will focus on systems in $D + 1$ spacetime dimensions which exhibit scaling anisotropic between time and space,

$$t \to b^z t, \qquad x^i \to b x^i, \qquad i = 1, \ldots D, \tag{1}$$

with the degree of anisotropy measured by the dynamical exponent z. Systems with such Lifshitz scaling appear frequently in quantum and statistical field theory of condensed matter systems [1], especially in the context of Lifshitz multicritical points, and in nonequilibrium statistical mechanics. More recently, in a seemingly unrelated development, anisotropic Lifshitz-type scaling (1) has played a central role in the new approach to quantum gravity initiated in Refs. [2,3] and commonly referred to as Hořava-Lifshitz (HL) gravity.

Since AdS/CFT correspondence taught us that many relativistic QFTs have relativistic gravity duals, it seems natural to expect that the two disparate applications of Lifshitz scaling—nonrelativistic QFT on one hand and HL gravity on the other—should similarly be related by a holographic duality. The background geometry that captures the spacetime symmetries of QFTs with Lifshitz scaling (1) is easy to find; it is given by the Lifshitz spacetime [4] in $D + 2$ dimensions,

$$ds^2 = -\left(\frac{r}{\ell}\right)^{2z} dt^2 + \left(\frac{r}{\ell}\right)^2 dx^i dx^i + \left(\frac{\ell}{r}\right)^2 dr^2. \tag{2}$$

(From now on, we will set its radius of curvature $\ell = 1$ for convenience.) The holographic gravity duals of Lifshitz-type QFTs should therefore have Eq. (2) as their solution.

Until now, the overwhelming share of work on Lifshitz holography (starting with Ref. [4]) does not use HL gravity; it uses relativistic bulk gravity coupled to matter instead. In the relativistic case, the coupling to matter is necessary, as the Lifshitz spacetime with $z \neq 1$ does not solve the Einstein equations in the vacuum. Here we stress that another natural option is available: Instead of adding *ad hoc* matter to Einstein gravity so that Eq. (2) becomes a solution, one can modify gravity itself.

Perhaps the most popular relativistic model for Lifshitz holography proposed in Ref. [5] consists of Einstein gravity [described by the bulk metric $G_{\mu\nu}$, in coordinates $y^\mu = (t, x^i, r)$, with $\mu = 0, \ldots D + 1$] coupled to a massive vector A_μ:

$$S_{\rm rel} = \frac{1}{16\pi G_{\rm N}} \int dt d^D x dr \sqrt{-G} \Big\{ \mathcal{R} - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \Big\} + \text{surface terms}. \tag{3}$$

The Lifshitz geometry (2) is a solution for an appropriate condensate of A_0 and an appropriate choice of Λ and m.

In this Letter, we will follow the alternate path and show that the Lifshitz spacetime is a vacuum solution of minimal HL gravity, with no additional matter. The preferred foliation of the Lifshitz spacetime, required for its embedding into HL gravity, is simply the foliation by leaves of constant t. We will often split the bulk coordinates y^μ into time t plus $D + 1$ spatial coordinates $y^a = (x^i, r)$, $i = 1, \ldots D$, and write the spacetime metric $G_{\mu\nu}$ in the Hamiltonian decomposition,

$$G_{\mu\nu} dy^\mu dy^\nu = -N^2 dt^2 + g_{ab}(dy^a + N^a dt)(dy^b + N^b dt).$$

Thus, g_{ab} is the metric on the spatial bulk leaves of fixed t, N_a is the shift vector, and N the lapse function. Since the Lifshitz geometry (2) requires N with a spatial dependence, we work in the nonprojectable version of HL gravity, with N a full-fledged spacetime-dependent field. Gauge symmetries are the foliation-preserving diffeomorphisms of spacetime.

HL gravity may enjoy better short-distance properties than Einstein gravity (if it is dominated at high energies by its own $z > 1$ scaling), but here we will follow the "bottom-up" strategy common in relativistic holography, and work only in the low-energy bulk gravity approximation. This is equivalent to the large N limit in the dual field theory. In this low-energy limit, HL gravity is dominated by the most relevant terms compatible with the gauge symmetries, and its effective action is

$$S = \frac{1}{2\kappa^2}\int dt d^D x dr \sqrt{g} N \left\{ K_{ab}K^{ab} - \lambda K^2 + \beta(R - 2\Lambda) + \frac{\alpha^2}{2}\frac{\nabla_a N \nabla^a N}{N^2} \right\}. \quad (4)$$

[Here $K_{ab} = \frac{1}{2N}(\partial_t g_{ab} - \nabla_a N_b - \nabla_b N_a)$ is the extrinsic curvature of the foliation, R the scalar curvature of g_{ab}, and $K = g^{ab}K_{ab}$.] The novelty compared to pure Einstein gravity is in the three couplings β, λ, and α, which in Einstein gravity are fixed to $\lambda = \beta = 1$ and $\alpha = 0$. Note that turning on the α coupling is important for the consistency of nonprojectable HL gravity [6,7]: Taking the naive $\alpha \to 0$ limit in Eq. (4) would lead to a nonclosure of the constraint algebra.

When $\Lambda = 0$, the flat spacetime $\mathbf{R}^{D+2}$ is a solution of Eq. (4). The propagating graviton modes consist of the transverse-traceless tensor polarizations with dispersion relation $\omega^2 = \beta k^2$ (here $k \equiv \sqrt{k_a k_a}$ is the magnitude of the spatial momentum), plus an extra scalar graviton polarization, with dispersion

$$\omega^2 = \frac{\beta(1-\lambda)}{[1-(D+1)\lambda]}\left[1 + D\left(\frac{2\beta}{\alpha^2} - 1\right)\right]k^2. \quad (5)$$

The requirement of stability and perturbative unitarity around flat spacetime constrains the couplings to be in the range $\beta > 0$,

$$\alpha^2 \leq \frac{2\beta D}{D-1}, \quad (6)$$

and

$$\lambda \geq 1 \quad \text{or} \quad \lambda \leq 1/(D+1). \quad (7)$$

Turning on the cosmological constant $\Lambda < 0$, we find that the Lifshitz geometry (2) is a vacuum solution of HL gravity with low-energy effective action (4), if

$$\Lambda = -\frac{(D+z-1)(D+z)}{2} \quad (8)$$

and

$$\alpha^2 = \frac{2\beta(z-1)}{z}. \quad (9)$$

This simple interpretation of Lifshitz spacetimes as vacuum solutions of HL gravity suggests that the latter is the natural minimal holographic model of holographic duality for Lifshitz-type field theories.

Further evidence for the universality of this minimal model of Lifshitz holography comes from the analysis of anisotropic Weyl anomalies in holographic renormalization initiated in Ref. [8] (and also later in Ref. [9]). Just like their relativistic counterparts, anisotropic Weyl anomalies contain a lot of universal information about the system, and serve as useful probes of the duality. Consider again some Lifshitz-type QFT on $\mathbf{R}^{D+1}$ with coordinates (t, x^i) and a general background metric

$$ds^2 = -\hat{N}^2 dt^2 + \hat{g}_{ij}(dx^i + \hat{N}^i dt)(dx^j + \hat{N}^j dt).$$

It is useful to think of this theory as residing at the spacetime boundary $r \to \infty$ of the $D+2$ dimensional asymptotically Lifshitz spacetime, with $\hat{g}_{ij}$, $\hat{N}_i$, and $\hat{N}$ being the components of the appropriately defined (anisotropic conformal class of the) boundary metric [8]. Generally, we will put hats on quantities defined at the boundary to distinguish them from their bulk counterparts. We define the anisotropic Weyl transformations generated by a spacetime-dependent $\sigma(t, x^i)$ as

$$\delta\hat{g}_{ij} = 2\sigma\hat{g}_{ij}, \qquad \delta\hat{N}_i = 2\sigma\hat{N}_i, \qquad \delta\hat{N} = z\sigma\hat{N}. \quad (10)$$

These represent a local generalization of the rigid scaling transformations (1). A QFT which is classically invariant under Eq. (10) can develop an anisotropic Weyl anomaly at the quantum level, with the effective action transforming as (see Appendix C of Ref. [8] for details)

$$\delta S_{\text{eff}}[\hat{g}_{ij}, \hat{N}_j, \hat{N}] = \int dt d^D x \sqrt{\hat{g}}\hat{N}\sigma(t, x^i)\mathcal{A}(t, x^i).$$

The independent terms that can appear in $\mathcal{A}(t, x^i)$ are local functionals of the metric, invariant under foliation preserving diffeomorphisms. They can be classified by solving a cohomological problem [8] designed to automatically incorporate the Wess-Zumino consistency conditions on the anomaly. However, the multiplicative coefficients with which these terms contribute to the anomaly (and which we will refer to as "central charges") must be calculated for a given theory on a case-by-case basis. Perhaps the simplest nontrivial case is $D = 2$ and $z = 2$. In this case, the anomaly is [8]

$$\mathcal{A} = c_K\left(\hat{K}_{ij}\hat{K}^{ij} - \frac{1}{2}\hat{K}^2\right) + c_V\left(\hat{R} - \frac{\hat{\nabla}^i\hat{N}\hat{\nabla}_i\hat{N}}{\hat{N}^2} + \frac{\hat{\Delta}\hat{N}}{\hat{N}}\right)^2, \quad (11)$$

with two independent central charges, c_K and c_V. (Here $\hat{K}_{ij}$, $\hat{\nabla}_i$, and $\hat{R}$ are the extrinsic curvature, connection, and the scalar curvature constructed from $\hat{g}_{ij}$). As noted in Ref. [8], $\mathcal{A}$ in Eq. (11) takes the form of the Lagrangian for $z = 2$ conformal HL gravity [2,3] in $2 + 1$ dimensions. Moreover, while the first term in Eq. (11) satisfies the so-called "detailed balance condition," (i.e., it is related to the square of the variation of another local functional W, see Refs. [2,3]), the other does not.

For QFTs with holographic gravity duals, we can calculate the anomaly by performing holographic renormalization of the bulk theory [10–12] (see Refs. [13–15] for reviews). The relativistic Weyl anomaly was calculated this way in Ref. [10]. Holographic renormalization for the relativistic bulk theory (3) in asymptotically Lifshitz spacetimes was developed and applied to the anisotropic Weyl anomalies in Ref. [8] following the earlier work of Refs. [16–19]. This procedure relies substantially on the notion of anisotropic conformal infinity developed in Ref. [20]. In the low-energy gravity approximation $S_{\rm eff}[\hat{g}_{ij}, \hat{N}_i, \hat{N}]$ is calculated by evaluating the on-shell gravity action with the appropriate falloff conditions on the metric field as $r \to \infty$. This on-shell action is divergent due to infinite volume and needs to be renormalized. We regulate it by cutting r off at $r_\epsilon = 1/\epsilon$, and expand the on-shell action asymptotically to reveal the structure of its divergences. For the special case of $D = z$, this expansion gives [8] (modulo terms that vanish as $\epsilon \to 0$)

$$\int dt d^D x \sqrt{\hat{g}} \hat{N} \left\{ \sum_{n=0}^{D-1} \frac{\mathcal{L}^{(2n)}}{\epsilon^{2(D-n)}} - \tilde{\mathcal{L}}^{(2D)} \log \epsilon + \mathcal{L}^{(2D)} \right\}.$$

(The $\log \epsilon$ term is present only for special values of D and z [8], including the case $D = z = 2$ of interest here.) The divergent terms are then canceled by local counterterms, and $S_{\rm eff} = \int dt d^D x \sqrt{\hat{g}} \hat{N} \mathcal{L}^{(2D)}$.

To calculate these divergent terms, we use the Hamiltonian form of holographic renormalization, as developed for relativistic AdS/CFT in Refs. [21,22] and extended to the asymptotically Lifshitz case in Ref. [8] following [17]. In this formulation, the on-shell bulk action is determined as a functional of the boundary metric because it satisfies the Hamilton-Jacobi equation for the radial evolution along r. The operator of radial evolution $\delta_{\mathcal{D}}$ is given by the generator of anisotropic dilatations on the boundary, and the Hamilton-Jacobi equation yields a recursive relation between the counterterms $\mathcal{L}^{(m)}$ of adjacent scaling dimensions m. One of these recursive relations implies [8]

$$\delta_{\mathcal{D}} \mathcal{L}^{(2D)} = -2D \mathcal{L}^{(2D)} + \tilde{\mathcal{L}}^{(2D)}. \qquad (12)$$

Interpreted from the boundary point of view, this means that when $\tilde{\mathcal{L}}^{(2D)} \neq 0$, $S_{\rm eff}$ scales anomalously under the $z = D$ anisotropic Weyl transformations, and $\tilde{\mathcal{L}}^{(2D)}$ is the anisotropic Weyl anomaly.

For the special case of the relativistic model (3) with $D = z = 2$, the divergent terms were calculated in Ref. [8], where we obtained for the anisotropic Weyl anomaly

$$\mathcal{A} \equiv \tilde{\mathcal{L}}^{(4)} = \frac{1}{16\pi G_{\rm N}} \left(\hat{K}_{ij} \hat{K}^{ij} - \frac{1}{2} \hat{K}^2 \right). \qquad (13)$$

Thus, the anomaly in this relativistic model turns out to have $c_V = 0$, or in other words, it satisfies the detailed balance condition. Why is it so? The conclusive answer was found in Ref. [8]: The relation implying that the anomaly in the relativistic model (3) should satisfy detailed balance follows from the holographic recursion relation between the counterterms $\mathcal{L}^{(2)}$ and $\tilde{\mathcal{L}}^{(4)}$, with $\mathcal{L}^{(2)}$ effectively playing the role of the local functional W (see Ref. [8] for details).

Looking for holographic duals of more general QFTs with both central charges independently nonzero is an interesting challenge. Before we embark on this pursuit, we should first check that QFTs whose central charges c_K and c_V are both nonzero indeed exist. Examples of strongly coupled Lifshitz field theories are very scarce to say the least, but our point can be made by considering the theory of the free $z = 2$ Lifshitz scalar Φ. When Φ is minimally coupled to background HL gravity,

$$S_\Phi = \int dt d^2 x \sqrt{\hat{g}} \left\{ \frac{1}{\hat{N}} (\partial_t \Phi - \hat{N}^i \hat{\nabla}_i \Phi)^2 - \hat{N} (\hat{\Delta} \Phi)^2 \right\},$$

this theory is classically invariant under Eq. (10) (with $\delta\Phi = 0$), but develops an anisotropic Weyl anomaly at the quantum level. This anomaly was calculated in Ref. [9], and it turns out to have $c_V = 0$. One could perhaps speculate that $c_V = 0$ might be a universal property of all consistent QFTs, hence eliminating the need for finding gravity duals with $c_V \neq 0$. A simple counterexample comes from coupling Φ to background gravity nonminimally, adding

$$-e^2 \int dt d^D x \sqrt{\hat{g}} \hat{N} \left\{ \hat{R} - \frac{\nabla^i \hat{N} \nabla_i \hat{N}}{\hat{N}^2} + \frac{\Delta \hat{N}}{\hat{N}} \right\}^2 \Phi^2$$

to S_Φ. Even with this nonminimal coupling, this theory stays classically invariant under the anisotropic Weyl transformations (again with $\delta\Phi = 0$), and develops a quantum anomaly. We calculated this anisotropic Weyl anomaly using the ζ-function regularization, and found $c_K = 1/(32\pi)$ and $c_V = -e^2/(8\pi)$.

Having demonstrated the existence of QFTs with $c_V \neq 0$, we can now ask how to reproduce this second central charge in a holographic gravity dual. One could look for relativistic models more complicated than Eq. (3) (see Ref. [23] for first results). Instead, we will show that minimal HL gravity already accounts for both of the independent central charges c_K and c_V in the anisotropic Weyl anomaly. In order to show that, we have performed holographic renormalization of Lifshitz spacetimes in pure HL gravity. Since the technicalities are quite involved (as they were in the relativistic model (3) studied in Ref. [8]), here we only present our main results; all technical details will appear in Ref. [24].

We find modified recursion relations for the divergent terms in the regulated action. In the special case $D = z = 2$, we solved these recursion relations and found that the logarithmic term $\tilde{\mathcal{L}}^{(4)}$ is equal to

$$\frac{1}{2\kappa^2} \left(\hat{K}_{ij} \hat{K}^{ij} - \frac{1}{2} \hat{K}^2 \right) + \frac{\beta}{48\kappa^2} \left(\hat{R} - \frac{\hat{\nabla}^i \hat{N} \hat{\nabla}_i \hat{N}}{\hat{N}^2} + \frac{\hat{\Delta} \hat{N}}{\hat{N}} \right)^2.$$

This is the anisotropic Weyl anomaly in our minimal model of Lifshitz holography with vacuum HL gravity. It is indeed of the most general form, with the two independent central charges given in terms of two low-energy couplings in minimal HL gravity: $c_K = 1/(2\kappa^2)$ and $c_V = \beta/(48\kappa^2)$.

The remaining coupling λ does not appear in the anomaly, but it still plays an important physical role. Just as in the case of flat spacetime, λ enters into the dispersion relation of the extra polarization of the graviton in the bulk. For example, in the radial gauge $g_{ir} = 0$, $g_{rr} = 1/r^2$, the extra graviton mode ϕ is found as the linearized fluctuation of the radial component of the shift vector, $N_r = \phi/r$. Returning to the case of general D and z, the linearized equations of motion imply the asymptotic behavior near infinity $\phi(r) \sim r^{D_\pm}$, with

$$D_\pm = \frac{1}{2}\left\{z - D \pm \sqrt{(z+D)^2 + \frac{4D(1-z)}{1-\lambda}}\right\}. \quad (14)$$

Standard rules of holographic duality will map $D_\pm$ to the scaling dimensions $\Delta_\pm$ of the operator dual to the extra graviton. Unitarity of the dual field theory requires that the scaling dimensions be real, implying (for $z > 1$)

$$\lambda \geq 1 \quad \text{or} \quad \lambda \leq \lambda_U \equiv \frac{(D-z)^2 + 4D}{(D+z)^2}. \quad (15)$$

This unitarity bound represents an intriguing analog of the Breitenlohner-Freedman bound familiar from relativistic holography: In HL gravity, the unitarity bound (15) allows the coupling λ to dip into the region between $1/(D+1)$ and 1, which according to Eq. (7) would be forbidden around flat spacetime. In the particularly interesting case of $D = z$, we get $\lambda_U = 1/D$, which opens up the previously forbidden regime $1/(D+1) \leq \lambda \leq 1/D$.

Now that we have seen that HL gravity provides candidate holographic duals for QFTs with anisotropic Lifshitz scaling, is it possible to apply HL gravity also to QFTs with isotropic $z = 1$ scaling? Interestingly, the limit $z \to 1$ corresponds to $\alpha \to 0$, the "unhealthy reduction" [6] of nonprojectable HL gravity, and may therefore be difficult to make sense of. This is perhaps to be expected: $z = 1$ QFTs with such gravity duals would likely exhibit isotropic dilatation symmetry without full relativistic conformal symmetry, a phenomenon whose examples are few and far between. Further study of our holographic duality in the $\alpha \to 0$ limit may shed new light on this rare class of QFTs.

Finally, throughout this Letter we have used the effective low-energy limit of HL gravity dominated by the terms of the lowest dimension in the action. We have been agnostic about how the model is completed at high energies. This completion may come from additional degrees of freedom (perhaps via an embedding into string theory) or it can be via a self-completion of HL gravity, due to highly anisotropic scaling at short distances. This latter possibility would be particularly interesting, as it could open a new door away from the large N limit and small bulk curvature. As this Letter was being finalized, complementary results about another form of nonrelativistic holography with HL gravity were presented in Refs. [25,26]. Our results, and those of Refs. [25,26], thus provide further evidence for the picture proposed originally in Ref. [8] that the natural arena for nonrelativistic holography is nonrelativistic HL gravity. It remains to be seen whether—as suggested in Ref. [8]—the nonrelativistic field theories whose holographic duals happen to be relativistic indeed represent only a minority among all theories with gravity duals.

We wish to thank Kevin Grosvenor, Omid Saremi, and Kostas Skenderis for useful discussions. This work has been supported by NSF Grants No. PHY-0855653 and No. PHY-1214644, DOE Award No. DE-AC02-05CH11231, and BCTP.

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