

Fundamental Bound on the Reliability of Quantum Information Transmission

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Information theory tells us that if the rate of sending information across a noisy channel were above the capacity of that channel, then the transmission would necessarily be unreliable. For classical information sent over classical or quantum channels, one could, under certain conditions, make a stronger statement that the reliability of the transmission shall decay exponentially to zero with the number of channel uses, and the proof of this statement typically relies on a certain fundamental bound on the reliability of the transmission. Such a statement or the bound has never been given for sending quantum information. We give this bound and then use it to give the first example where the reliability of sending quantum information at rates above the capacity decays exponentially to zero. We also show that our framework can be used for proving generalized bounds on the reliability.

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Introduction.—The capacity of a given channel is defined as the highest rate of sending information (measured as the amount of information sent per channel use) reliably in the limit of a large number of channel uses [1–3]. The converse of the channel capacity theorem tells us that sending information at rates higher than the capacity would necessarily be unreliable. A strong converse additionally tells us that the reliability would be very small and, in some cases more explicitly, would decay exponentially to zero with the number of channel uses. Not all channels have a strong converse [4].

Such strong converses are available for sending classical information across classical or quantum channels (under certain conditions) and are typically shown using a fundamental bound on the reliability. But, somewhat surprisingly, there has been no such strong converse when quantum information is sent across a quantum channel and an equivalent bound has been unknown. We first prove this bound in full generality and then apply it to give the first example of a strong converse for quantum information transfer where the reliability decays exponentially to zero with the number of channel uses.

A strong converse establishes capacity as a sharp threshold for information transmission and is clearly of great theoretical interest. It also has interesting applications in cryptography. Let Alice have an unlimited noise-free quantum memory to store qubits while Bob has a noisy quantum memory (also called the noisy-storage assumption). If the strong converse holds for the quantum channel modeling the noise that acts on Bob’s memory, then Alice and Bob can implement any two-party cryptographic task securely [5].

We now provide a more detailed but high level overview of our results. A protocol to transfer information (classical or quantum) across a noisy communication channel is characterized by the amount of information (\mathcal{R}) it conveys and the reliability (\mathbb{F}) it promises. Typical definitions of

reliability ensure that $\mathbb{F} \in [0, 1]$, where $\mathbb{F} \approx 1$ would imply a highly reliable information transfer; i.e., information sent and reconstructed at the receiver is very close to each other ($\mathbb{F} = 1$ implies an exact match) and $\mathbb{F} \approx 0$ would imply a highly unreliable transmission.

Information could be classical or quantum. Classical information is an unknown sequence of bits (such as an email message) that Alice wants to send to Bob. A quantum information transfer can also be looked upon as entanglement transfer [3]. Alice has a quantum system S (information) that is entangled with a reference system A and Alice (who doesn’t have access to A) wishes to send a quantum system through a noisy environment (that doesn’t act upon A) such that at the end of the protocol, the state of A and Bob’s system (say \hat{S}) is close to the state of A and S .

The fundamental bound that we seek for all $s \in [-\beta, 0)$ and protocol parameters α is given by

$$\mathbb{F} \leq e^{s\mathcal{R} - E_0(s, \alpha)}, \quad (1)$$

where $E_0(0, \alpha) = 0$, the derivative of $E_0(s, \alpha)$ with respect to s at $s = 0$ gives us a measure of information that could be transferred across the channel reliably, and β is a constant independent of α and \mathcal{R} that, for our purposes, is 0.5.

$E_0(s, \alpha) - s\mathcal{R}$ is known as the Gallager’s exponent named after R. G. Gallager, who first proposed it in a different setting [6]. The bound in Eq. (1) was shown for classical information sent across a classical channel (Arimoto [7]) and a quantum channel (Ogawa and Nagaoka [8]). Winter gave another proof of the strong converse for sending classical information over quantum channels without the Gallager’s exponent [9]. Extensions of the above results are due to König and Wehner (Ref. [10]) and further upper bounds to fidelity for entanglement unassisted and assisted codes are given by Matthews and Wehner [11].

The search for a quantum Gallager's exponent when quantum information is sent across a quantum channel has been a long-standing problem and we provide it in this Letter. Table I lists these various cases.

Our proof relies on using the monotonicity property (mentioned below) satisfied by many information divergences. The idea of proving bounds on the reliability for classical protocols using monotonicity dates back to Blahut's work [12] and has been used further more recently [13–15].

We now provide a brief and heuristic explanation as to why this bound is considered fundamental. Let us define for a single use of the channel that $I(\boldsymbol{\alpha}) = \partial E_0(s, \boldsymbol{\alpha}) / \partial s|_{s=0}$ and $C = \max_{\boldsymbol{\alpha}} I(\boldsymbol{\alpha})$ is called the channel capacity, where $\boldsymbol{\alpha}'$ is the part of $\boldsymbol{\alpha}$ that can be changed by fine-tuning the protocol [1–3]. There are parameters in the setup that cannot be changed, such as the channel, and there may be some practical constraints, such as the energy used for transmission, that the protocol must obey. Because E_0 obeys $E_0(0, \boldsymbol{\alpha}) = 0$, for a negative s near 0, $-E_0(s, \boldsymbol{\alpha}) \approx -sI(\boldsymbol{\alpha}) \leq -sC$ and the above bound could be weakened to give $\mathbb{F} \leq e^{s(\mathcal{R}-C)}$. Hence, if $\mathcal{R} > C$, then \mathbb{F} is always exponentially bounded away from 1. If we use the channel n times for sending $n\mathcal{R}$ amount of information, then we could, under certain conditions, write the above bound as $\mathbb{F} \leq e^{sn(\mathcal{R}-C)}$. If $\mathcal{R} > C$, then $\mathbb{F} \rightarrow 0$ exponentially with n ; i.e., if we are pumping information into the channel higher than the capacity, then the transmission would be quite unreliable.

We shall frequently deal with the quantum Rényi divergences in this Letter that for parameter $\lambda \geq 0$ are given by

$$D_\lambda(\rho \parallel \sigma) = \frac{1}{\lambda - 1} \ln \text{Tr} \rho^\lambda \sigma^{1-\lambda}, \quad (2)$$

where the limit is taken at $\lambda = 1$. We shall confine ourselves with $\lambda \in (1, 2]$ in this Letter and deal with finite dimensional quantum systems. The following two properties are needed later.

Property 1.—It has been shown (see Example 4.5 in Ref. [16]) that for the chosen range of λ , D_λ satisfies the monotonicity property; i.e., for any two un-normalized density matrices (that are positive but need not have a unit trace) ρ , σ and a completely positive and trace preserving (CPTP) quantum operation \mathcal{N} acting on them, we have

$$D_\lambda(\rho \parallel \sigma) \geq D_\lambda[\mathcal{N}(\rho) \parallel \mathcal{N}(\sigma)]. \quad (3)$$

TABLE I. Gallager's exponent (that gives an exponential upper bound on reliability) for various cases.

Information	Channel	Proposed by
Classical	Classical	Arimoto [7]
Classical	Quantum	Ogawa and Nagaoka [8]
Quantum	Quantum	(this Letter)

Property 2.—We shall also need the following queer property that is not difficult to prove. Let $\Pi_0 = |0\rangle\langle 0|$ and $\Pi_1 = |1\rangle\langle 1|$ be two projectors with $\Pi_0 + \Pi_1 = \mathbb{1}$. Let $\alpha \in [0, 1]$, $\beta \in (0, 1]$, $\rho = \alpha\Pi_0 + (1 - \alpha)\Pi_1$, $\sigma = \beta\Pi_0 + (1/\beta - \beta)\Pi_1$, and let us define

$$\mathbb{D}_\lambda(\alpha \parallel \beta) := D_\lambda(\rho \parallel \sigma). \quad (4)$$

Note that $\sigma \geq 0$ but does not have unit trace. Then $\mathbb{D}_\lambda(\alpha \parallel \beta)$ is independent of the choice of $\{\Pi_0, \Pi_1\}$ and increasing for all $\alpha \geq \beta$.

We now derive a quantity from the Rényi divergence as

$$K_\lambda(A)B)_\rho := \inf_{\sigma^B \in \mathcal{S}(\mathcal{H}_B)} D_\lambda(\rho^{AB} \parallel \mathbb{1} \otimes \sigma^B), \quad (5)$$

where \mathcal{H}_B is the Hilbert space describing quantum system B and $\mathcal{S}(\mathcal{H}_B)$ is the set of all density matrices of \mathcal{H}_B , and $\mathbb{1}$ is the identity matrix whose dimensions should be clear from the context. Csiszár defined a similar quantity in the classical case and related it to the Gallager's exponent [17]. The following properties of $K_\lambda(A)B)_\rho$ will be useful later.

Lemma 1.—Let $\mathcal{E}^{B \rightarrow C}$ be a quantum operation and $\rho^{AC} = \mathcal{E}^{B \rightarrow C}(\rho^{AB})$. Then

$$K_\lambda(A)B)_\rho \geq K_\lambda(A)C)_\rho.$$

Lemma 2.—Let $\rho^{AA'}$ be any quantum state in AA' , and $\rho^{AB} = \mathcal{N}^{A' \rightarrow B}(\rho^{AA'})$. Then

$$K_\lambda(A)B)_\rho = \frac{\lambda}{1 - \lambda} E_0(\lambda^{-1} - 1, \mathcal{N}^{A' \rightarrow B})_\rho,$$

where for $s = \lambda^{-1} - 1$,

$$E_0(s, \mathcal{N}^{A' \rightarrow B})_\rho := -\ln \text{Tr}[\text{Tr}_A(\rho^{AB})^{1/(s+1)}]^{s+1}.$$

Information processing task.—Suppose a quantum system S and a reference system A have a state $|\phi\rangle^{AS}$. Alice only has access to the system S and not to A . Alice wants to send her part of the shared state with A to Bob using n independent uses of a quantum channel $\mathcal{N}^{A' \rightarrow B}$ such that at the end of the communication protocol chain, Bob's shared state with the reference A is arbitrarily close to the state Alice shared with A . We shall call \mathcal{R} the communication rate given by $\mathcal{R} := \ln |S|/n$, where $|S|$ is the dimension of \mathcal{H}_S . We shall assume that the state of S is given by $\mathbb{1}/|S|$, i.e., the completely mixed state.

To this end, Alice performs an encoding operation given by $\mathcal{E}^{S \rightarrow A^n}$ to get $\rho^{AA^n} = \mathcal{E}^{S \rightarrow A^n}(\phi^{AS})$. Alice transmits the system A^n over $\mathcal{N}^{A^n \rightarrow B^n} = (\mathcal{N}^{A' \rightarrow B})^{\otimes n}$ and Bob receives the state $\rho^{AB^n} = \mathcal{N}^{A^n \rightarrow B^n}[\mathcal{E}^{S \rightarrow A^n}(\phi^{AS})]$. Bob applies a decoding operation on his part of the received state to get $\rho^{A\hat{S}} = \mathcal{T}^{B^n \rightarrow \hat{S}}\{\mathcal{N}^{A^n \rightarrow B^n}[\mathcal{E}^{S \rightarrow A^n}(\phi^{AS})]\}$. The performance of the protocol is quantified by the fidelity given by $F(\phi^{AS}, \rho^{A\hat{S}}) = \langle \phi |^{AS} \rho^{A\hat{S}} | \phi \rangle^{AS}$. If a protocol promises a fidelity not smaller than \mathbb{F} , then we shall refer to such a protocol as a $(n, \mathcal{R}, 1 - \mathbb{F})$ code.

The maximum rate per channel use for this protocol in the limit of a large number of channel uses and fidelity arbitrarily close to 1 was proved in a series of papers (see Refs. [18–25]). Let the coherent information of the channel $\mathcal{N}^{A' \rightarrow B}$ be defined as $Q(\mathcal{N}) := \max_{\rho^{AA'}} I(A)B)_\sigma$, where $\sigma^{AB} = \mathcal{N}^{A' \rightarrow B}(\rho^{AA'})$, $I(A)B)_\sigma := H(B)_\sigma - H(A, B)_\sigma$, and $H(A)_\sigma$ is the von Neumann entropy of a quantum state σ in a system A given by $H(A)_\sigma = -\text{Tr} \sigma \ln \sigma$. The capacity of the channel is now given by the regularization $Q_{\text{reg}}(\mathcal{N}) := \lim_{n \rightarrow \infty} Q(\mathcal{N}^{\otimes n})/n$.

We now prove an inequality involving the fidelity and the rate.

Theorem 1.—For $\mathbb{F} \geq e^{-n\mathcal{R}}$, any $(n, \mathcal{R}, 1 - \mathbb{F})$ code satisfies

$$\mathbb{D}_\lambda(\mathbb{F} \parallel e^{-n\mathcal{R}}) \leq K_\lambda(A)B^n)_\rho.$$

Proof.—Let $\{|i\rangle^{AS}\}$ be an orthonormal basis for \mathcal{H}^{AS} with $|1\rangle^{AS} = |\phi\rangle^{AS}$. Consider a CPTP quantum map $\mathcal{F}^{AS \rightarrow C}$ where $|C| = 2$ with Kraus operators $|0\rangle^C \langle 1|^{AS}$, and $\{|1\rangle^C \langle i|^{AS}\}$, $i = 2, 3, \dots, |AS|$. Let $\Pi_0^C = 0^C$ and $\Pi_1^C = 1^C$. Then for all $\sigma^{\hat{S}}$, we have $\mathcal{F}(\rho^{A\hat{S}}) = \mathbb{F}' \Pi_0^C + (1 - \mathbb{F}') \Pi_1^C$, $\mathcal{F}(\mathbb{1} \otimes \sigma^{\hat{S}}) = e^{-n\mathcal{R}} \Pi_0^C + (e^{n\mathcal{R}} - e^{-n\mathcal{R}}) \Pi_1^C$, where $\mathbb{F}' = \langle \phi | \rho^{AS} | \phi \rangle$. We now have the following inequalities

$$\begin{aligned} K_\lambda(A)B^n)_\rho &\stackrel{a}{\geq} \inf_{\sigma^{\hat{S}}} D_\lambda(\rho^{A\hat{S}} \parallel \mathbb{1} \otimes \sigma^{\hat{S}}) \\ &\stackrel{b}{\geq} \inf_{\sigma^{\hat{S}}} D_\lambda[\mathbb{F}' \Pi_0^C + (1 - \mathbb{F}') \Pi_1^C \parallel e^{-n\mathcal{R}} \Pi_0^C \\ &\quad + (e^{n\mathcal{R}} - e^{-n\mathcal{R}}) \Pi_1^C] \stackrel{c}{=} \mathbb{D}_\lambda(\mathbb{F}' \parallel e^{-n\mathcal{R}}) \\ &\stackrel{d}{\geq} \mathbb{D}_\lambda(\mathbb{F} \parallel e^{-n\mathcal{R}}), \end{aligned}$$

where a and b follow from the data processing inequality and the definition of K_λ , c follows since the quantity $\mathbb{D}_\lambda(\mathbb{F}' \parallel e^{-n\mathcal{R}})$ is independent of $\sigma^{\hat{S}}$, and d from the Property 2 of \mathbb{D}_λ . \square

The constraint $\mathbb{F} \geq e^{-n\mathcal{R}}$ may not be seen as weakening the bound because, if the constraint is violated, i.e., $\mathbb{F} \leq e^{-n\mathcal{R}}$, then this, by itself, would imply an exponential convergence of \mathbb{F} to 0. We first note that

$$\mathbb{D}_\lambda(\mathbb{F} \parallel e^{-n\mathcal{R}}) \geq \frac{\lambda}{\lambda - 1} \ln \mathbb{F} + n\mathcal{R} \quad (6)$$

and it follows from Lemma 2 and Theorem 1 that

$$\mathbb{F} \leq e^{s n \mathcal{R} - E_0[s, (\mathcal{N}^{A' \rightarrow B})^{\otimes n}]_\rho}, \quad (7)$$

which gives us the quantum Gallager's exponent. The properties of E_0 are studied by the following theorem.

Theorem 2.—For any quantum state σ^{AB} , $s \in [-1/2, 0)$, the function

$$g(s) := -\ln \text{Tr}[\text{Tr}_A(\sigma^{AB})^{1/(s+1)}]^{s+1},$$

satisfies

$$g(0) = 0, \quad \left. \frac{\partial g(s)}{\partial s} \right|_{s=0} = I(A)B)_\sigma,$$

and $g(s) + (s + 1) \ln |A|$ is an increasing function in s .

We note here that only the two above mentioned properties of the quantum Rényi divergence are used for our results. Hence, if the Rényi divergence is replaced by any other divergence that satisfies these two properties, then Theorem 1 shall hold for that divergence as well. The noncommutative hockey-stick divergence that we now define is one such example that for $\rho, \sigma \geq 0$, and $\gamma \geq 1$ is given by $\mathcal{D}(\rho \parallel \sigma) = \text{Tr}(\rho - \gamma\sigma)^+$, where κ^+ is the positive part of a Hermitian matrix $\kappa = \kappa^+ - \kappa^-$, $\kappa^+, \kappa^- \geq 0$. It can be regarded as a noncommutative generalization of the classical f -relative entropy (see Ref. [26]) using the hockey-stick function $f(x) = (x - \gamma)^+$ [27]. We similarly define a derived quantity as

$$\mathcal{K}(A)B)_\rho := \inf_{\sigma^B \in \mathcal{S}(\mathcal{H}_B)} \mathcal{D}(\rho^{AB} \parallel \mathbb{1} \otimes \sigma^B).$$

Quantum erasure channel with maximally entangled inputs.—We show that the fidelity would decrease exponentially with the number of channel uses for rates above the capacity for maximally entangled inputs that have the full Schmidt rank.

A quantum erasure channel transmits the input state with probability $1 - p$ and “erases” it, i.e., replaces it with an orthogonal erasure state with probability p [28] (see also Ref. [29]). The dimension of the output Hilbert space is one larger than that of the input.

A quantum erasure channel $\mathcal{N}_p^{A' \rightarrow B}$, defined in Ref. [3], is given by the following Kraus operators $\{\sqrt{(1-p)} \sum_{i=1}^{|A'|} |i\rangle^B \langle i|^{A'}, \sqrt{p} |e\rangle^B \langle 1|^{A'}, \dots, \sqrt{p} |e\rangle^B \langle |A'|^{A'}\}$, $i = 1, \dots, |A'|$, $p \in [0, 1]$, $|B| = |A'| + 1$, $\{|i\rangle^{A'}\}$, $\{|i\rangle^B\}$ are orthonormal bases in $\mathcal{H}_{A'}$ and \mathcal{H}_B , respectively, and $|e\rangle^B = |j\rangle^B$ for $j = |B|$. The action of the channel can be understood as follows

$$\mathcal{N}_p^{A' \rightarrow B}(\rho^{AA'}) = (1 - p)\sigma^{AB} + p\rho^A \otimes |e\rangle\langle e|^B.$$

Let $\sigma^{AB} = \mathcal{G}^{A' \rightarrow B}(\rho^{AA'})$, where \mathcal{G} increases the dimension but leaves the state intact. Then with probability $1 - p$, the channel leaves the state as σ^{AB} and with probability p , it erases the state and replaces it with $|e\rangle^B$. It is not difficult to see that σ^{AB} is orthogonal to $\rho^A \otimes |e\rangle\langle e|^B$.

Taking this further for n channel uses, let $\sigma^{AB^n} = (\mathcal{G}^{A' \rightarrow B})^{\otimes n}(\rho^{AA^n})$. The output can be written as the sum of 2^n orthogonal density matrices where each of these matrices results from i erasures $i \in \{0, \dots, n\}$ and this occurs with probability $(1 - p)^{n-i} p^i$. The number of states that have suffered exactly i erasures is $\binom{n}{i}$.

Let $B_{i_1} \cdots B_{i_{n-k}}$ be the quantum systems that have not suffered erasures and we could write the state in this case using σ^{AB^n} as

$$\zeta_{i_1, \dots, i_{n-k}}^{AB_{i_1 \dots B_{i_n}}} = \sigma^{AB_{i_1 \dots B_{i_{n-k}}} \otimes \bigotimes_{j=1}^k |e\rangle\langle e|^{B_{i_{n-k+j}}}.$$

It now follows that

$$\rho^{AB^n} = \sum_{2^n \text{ terms}} \alpha_{k,n} \times \zeta_{i_1, \dots, i_{n-k}}^{AB_{i_1 \dots B_{i_n}}}, \quad (8)$$

where $\alpha_{k,n} = (1-p)^{n-k} p^k$.

To prove the strong converse, we find an upper bound for $K_\lambda(A)B^n$. We assume that ρ^{AA^n} is a maximally entangled state with a Schmidt rank of d_A^n where $d_A = |A'|$. Note that this is the capacity-achieving input for this channel and $Q(\mathcal{N}) = (1-2p)^+ \ln d_A$ is the single-letter quantum capacity for this channel [30] (see also Ref. [3]). Note that $d_A^k \times \rho^{AA'_1 \dots A'_{n-k}}$ is a projector of rank d_A^k and $\rho^{A'_1 \dots A'_{n-k}}$ is the maximally mixed state.

Theorem 3.—The strong converse holds for the quantum erasure channel for the above chosen maximally entangled channel inputs.

Proof.—Note the following set of inequalities for $s = \lambda^{-1} - 1$, $\lambda \in (1, 2]$

$$\begin{aligned} K_\lambda(A)B^n &\stackrel{a}{=} -\frac{1}{s} \ln \text{Tr}[\text{Tr}_A(\rho^{AB^n})^\lambda]^{1/\lambda} \\ &\stackrel{b}{=} -\frac{1}{s} \ln \sum_{2^n \text{ terms}} \alpha_{k,n} \text{Tr}[\text{Tr}_A(\zeta_{i_1, \dots, i_{n-k}}^{AB_{i_1 \dots B_{i_n}}})^\lambda]^{1/\lambda} \\ &\stackrel{c}{\leq} -\frac{1}{s} \ln \left\{ \sum_{2^n \text{ terms}} \alpha_{k,n} \exp \left[\frac{-K_\lambda(A)A'_{i_1} \dots A'_{i_{n-k}}}{s} \right] \right\}, \end{aligned}$$

where a follows from Lemma 2, b follows from Eq. (8) and the orthogonality of ζ 's and c follows because K_λ satisfies monotonicity and Lemma 2. Using the fact that $d_A^k \times \rho^{AA'_1 \dots A'_{n-k}}$ is a projector of rank d_A^k , we get $K_\lambda(A)B^n \leq nE_0(s)/s$ where we define (with some abuse of notation)

$$E_0(s) := -\ln[(1-p)d_A^{-s} + pd_A^s]$$

and $E_0(0) = 0$. Using (6), we have

$$\mathbb{F} \leq \exp\{n[s\mathcal{R} - E_0(s)]\}.$$

Furthermore, for $p \in [0, 1/2]$,

$$\lim_{s \downarrow 0} \frac{E_0(s)}{s} = Q(\mathcal{N}).$$

Hence, for all $\mathcal{R} > Q(\mathcal{N})$, $\exists s \in [-1/2, 0)$ such that $\mathcal{R} - E_0(s)/s > 0$, and thus the strong converse holds. For $p > 1/2$, $E'_0(0) < 0$ and, hence, using similar arguments as above, for any $\mathcal{R} > 0$, the strong converse holds. \square

An alternate proof of Theorem 3 using the hockey-stick divergence is provided in the Supplemental Material [31].

To summarize our results, we have given an exponential upper bound on the reliability of quantum information transmission. The bound is fundamental in the same vein as the bounds known for the transmission of classical information across classical or quantum channels (see

Refs. [7,8,10]) and holds under general conditions. We then apply our bound to yield the first known example for exponential decay of reliability at rates above the capacity for quantum information transmission.

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- [31] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.110.080501> for the proofs omitted in the main text.