Magnetic Fields in Superconducting Neutron Stars

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The interior of a neutron star is likely to be predominantly a mixture of superfluid neutrons and superconducting protons. This results in the quantization of the star's magnetic field into an array of thin flux tubes, producing a macroscopic force very different from the Lorentz force of normal matter. We show that in an axisymmetric superconducting equilibrium the behavior of a magnetic field is governed by a single differential equation. Solving this, we present the first self-consistent superconducting neutron star equilibria with poloidal and mixed poloidal-toroidal fields and also give the first quantitative results for the corresponding magnetically induced distortions to the star. The poloidal component is dominant in all our configurations. We suggest that the transition from normal to superconducting matter in a young neutron star may cause a large-scale field rearrangement.

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There is now compelling evidence that the bulk of a neutron star is composed of superfluid neutrons and superconducting protons. It has long been thought that neutron stars would be cold enough to contain these states of matter [1–3], based on expectations from the theory of terrestrial superconductivity [4]. A long-standing piece of evidence in favor of superfluidity is the phenomenon of pulsar glitches—sudden events in which the star spins *up* slightly [5]. The best explanation for the larger glitches is as a transfer of angular momentum from the more rapidly rotating superfluid to the crust. In addition, recent observations of the rapidly cooling young neutron star in the Cassiopeia A supernova remnant are well explained by the onset of neutron superfluidity together with proton superconductivity in the core [6,7], suggesting that these components will be present in a neutron star below a critical temperature of around 10^9 K. The most highly magnetized neutron stars, the magnetars, have high observed temperatures which might suggest that their transition to superfluidity is delayed. This is due to crustal heating, however, while the thermally decoupled magnetar core is expected to cool rapidly below the critical temperature [8]. Thus *all* observed neutron stars are likely to contain superfluid and superconducting components.

In terrestrial superconductors the Meissner effect expels any magnetic field below some critical strength, but in neutron stars the expulsion time scale is extremely long [2], so the magnetic field exists in a metastable state. Nonetheless, it is affected: In a large part of the star, the protons are expected to form a type-II superconductor, meaning that the field will be quantized into flux tubes surrounded by unmagnetized matter. On the macroscopic scale, this changes the nature of the magnetic force from the Lorentz force of normal matter to a flux tube tension force [9–11]. The innermost part of the core may exhibit type-I superconductivity [12,13], with large regions of alternating normal and superconducting matter; the effect of this on the global magnetic field is unknown at present. Alternatively, the star's hadronic matter could give way to an inner core of "color superconducting" quark matter [14,15].

A neutron star's magnetic field can play a variety of important roles. It affects the temperature and rotational evolution of the star—and rotation is the key observational feature used to determine the star's age. Understanding the interior fields of apparently different classes of neutron star could help unify them into a single canonical stellar model [16,17]. Magnetic field effects could explain the differing nature of glitches in pulsars and magnetars and postglitch recovery [18]. In magnetars, the magnetic field provides the energy that powers their giant flares and is important for understanding their observed quasiperiodic oscillations [19,20]. Finally, a magnetic field induces a distortion which will generally not be aligned with the star's rotation axis. This system will therefore produce gravitational waves [21], perhaps at an amplitude great enough for future detection.

Motivated by the above reasons, there has been a great deal of recent work on neutron star magnetic fields, but almost all of it is based on models assuming normally conducting matter. This may be partly because this case is more familiar, thanks to the large body of literature on nondegenerate stars [22]. For neutron stars, superconductivity is a key missing ingredient, whose inclusion is an essential step toward more realistic models.

In this Letter, we try to lay some foundations for the modeling of superconducting neutron stars. We study equilibrium configurations, motivated by the observation that neutron star magnetic fields appear to be long-lived, evolving only on long time scales. We show that in axisymmetry the magnetic field of a superconducting star is governed by a single differential equation. This is analogous to the Grad-Shafranov equation for normal matter [23,24] but more complicated: It involves terms related to the local field strength. We describe a method of solution for the equation and present the results. Other than the special case of a purely toroidal field [25,26], these are the first self-consistent solutions for magnetic fields in a superconducting star (see, however, the simplified poloidal-field model constructed in Ref. [27]). Since the submission of this Letter, a new study on poloidal fields has appeared [28]. We compare with the corresponding normal-matter results and discuss the implications for the internal magnetic fields of neutron stars.

Axisymmetric magnetic stars.—We model a neutron star as a three-fluid system, with superfluid neutrons, electrons, and type-II superconducting protons. The electrons have negligible inertia, however, and their chemical potential can be incorporated into that of the protons. Therefore we can reduce to a two-fluid system of equations, denoting neutron quantities with a subscript n and the combined proton and electron quantities with a p. We choose a rather idealized equation of state, a two-fluid analogue of a polytrope [29], so that the chemical potential $\tilde{\mu}_x$ of each species ($x = \{n, p\}$) is a function of the corresponding mass density ρ_x: $\tilde{\mu}_x = \tilde{\mu}_x(\rho_x)$. By working in cylindrical polar coordinates (ϖ, ϕ, z), the two Euler equations for our model may be written

$$\nabla\left(\tilde{\mu}_x + \Phi - \frac{\varpi^2\Omega_x}{2}\right) = \frac{\mathbf{F}_x}{\rho_x}, \tag{1}$$

where Φ denotes gravitational potential, Ω_x rotation rate, and $\mathbf{F}_x$ magnetic force. Although we consider only nonrotating models here, the following derivations and numerics follow through for cases with corotating neutrons and protons, $\Omega_n = \Omega_p$. In general, one would expect an *entrainment* effect, leading to coupling between neutrons and protons and to an effective magnetic force on the neutrons. We neglect this for simplicity, so that $\mathbf{F}_n = 0$, and write the proton force as $\mathfrak{F}_{\rm mag}$. The particle species are therefore coupled only through gravity,

$$\nabla^2\Phi = 4\pi G(\rho_n + \rho_p). \tag{2}$$

Now, taking the curl of the proton Euler equation, we see that there exists a scalar M such that

$$\frac{\mathfrak{F}_{\rm mag}}{\rho_p} = \nabla M, \tag{3}$$

regardless of whether the protons are normal or superconducting. Another universal result is that $\mathbf{B}$ must be divergence free; using this together with the assumption of axisymmetry allows us to write

$$\mathbf{B} = \frac{1}{\varpi}\nabla u \times \mathbf{e}_\phi + B_\phi \mathbf{e}_\phi, \tag{4}$$

which defines the stream function u. Note that $\mathbf{B}\cdot\nabla u = 0$; field lines are parallel to constant-u contours.

Normal matter.—In this familiar case, $\mathfrak{F}_{\rm mag}$ is the Lorentz force, given by

$$\mathfrak{F}_{\rm mag} = \frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B}. \tag{5}$$

It can be shown that the toroidal field component is governed by a function $f_N(u) = \varpi B_\phi$ [23,24]. Using this and the general results of the last section, one may derive the Grad-Shafranov equation,

$$\begin{aligned}\Delta_* u &\equiv \frac{\partial^2 u}{\partial\varpi^2} - \frac{1}{\varpi}\frac{\partial u}{\partial\varpi} + \frac{\partial^2 u}{\partial z^2} \\ &= -4\pi\rho_p\varpi^2\frac{dM}{du} - f_N\frac{df_N}{du}.\end{aligned} \tag{6}$$

This governs the form of a poloidal or mixed poloidal-toroidal field in axisymmetric equilibrium. The single-fluid version of this has been the basis of numerous studies of magnetic stars.

Superconducting matter.—By averaging the contribution of the flux tubes in a type-II superconductor, one arrives at a macroscopic expression for the magnetic force [9–11],

$$\mathfrak{F}_{\rm mag} = -\frac{1}{4\pi}\left[\mathbf{B}\times(\nabla\times\mathbf{H}_{c1}) + \rho_p\nabla\left(B\frac{\partial H_{c1}}{\partial\rho_p}\right)\right], \tag{7}$$

where $\mathbf{H}_{c1} = H_{c1}\hat{\mathbf{B}}$ is the microscopic critical field, $\mathbf{B}$ the smooth-averaged macroscopic field, and $\hat{\mathbf{B}} = \mathbf{B}/B$ the unit tangent vector to the magnetic field. In the absence of entrainment, $H_{c1} = h_c\rho_p$ to a good approximation, where h_c is a constant [11]. Using this relation and Eqs. (4) and (7), we get an expression for the magnetic force with various poloidal terms (see Ref. [26]) and one toroidal term, which must be zero by axisymmetry,

$$(\mathfrak{F}_{\rm mag})_\phi = \nabla u \times \nabla(\rho_p\varpi\hat{B}_\phi) = 0. \tag{8}$$

This gives us a dichotomy: Satisfying (8) leads to either a mixed poloidal-toroidal field or a purely toroidal field. For the latter case, we satisfy (8) by taking $\nabla u = \mathbf{0}$; this has been discussed in earlier work [25,26]. For the former, we require that ∇u and $\nabla(\rho_p\varpi\hat{B}_\phi)$ be parallel, which leads to

$$\rho_p\varpi\hat{B}_\phi = f(u) \tag{9}$$

for some function f. In the special case $f(u) = 0$, the field is purely poloidal.

One key step in the derivation of the Grad-Shafranov equation is showing that $M = M(u)$. In the superconducting case this is no longer true. We can, however, define a related function y which *is* a function of u,

$$y(u) = \frac{4\pi M}{h_c} + B. \tag{10}$$

Using the functions $y(u)$ and $f(u)$, together with the poloidal magnetic-force terms from Ref. [26], we arrive at a single differential equation governing the magnetic field,

$$\Delta_* u = \frac{\nabla\Pi \cdot \nabla u}{\Pi} - \varpi^2 \rho_p \Pi \frac{dy}{du} - \Pi^2 f \frac{df}{du}, \qquad (11)$$

where $\Pi \equiv B/\rho_p$. The above equation is the equivalent of the Grad-Shafranov equation when the protons form a type-II superconductor instead of being a normal fluid. The result for a single-fluid superconducting star may be obtained by replacing ρ_p with ρ, the total density, and is valid for barotropic equations of state. The most significant differences from the normal case are the presence of the Π factors and the fact that the magnetic force no longer appears explicitly through the function M.

Superconducting core and normal crust.—The superfluid and superconducting matter of a neutron star's core do not extend to the stellar surface; instead, the star has an elastic crust of normal matter. It is also numerically difficult to solve Eq. (11) matched directly to a potential field ($\nabla \times \mathbf{B} = 0$) exterior. For these reasons we choose a canonical neutron star model with a core of superfluid neutrons and type-II superconducting protons, matched to a single-fluid relaxed crust composed solely of normal protons; this in turn is matched to a vacuum exterior with potential field at the stellar surface R_*.

The complicated physics at a neutron star's crust-core boundary may well include the presence of a current sheet and a discontinuity in the magnetic field (Ref. [28] contains some discussion of this issue). In addition, the pinning of flux tubes to the crust could be important. Quantifying these effects is beyond the scope of this Letter, however, so for this first study we assume continuity of the magnetic field at the boundary. We also demand magnetic force balance, matching the smooth-averaged flux tube tension force and the Lorentz force at the crust-core interface. More specifically, we define the crust-core boundary as an isopycnic contour $\rho_p = \rho_p^{cc}$ (at a near-constant radius of $r = 0.9R_*$), requiring that the magnetic-force scalar function M be continuous and that $\mathbf{H}_{c1} \to \mathbf{B}$ there. This may be done by defining the superconducting functions $y(u)$ and $f(u)$ in terms of their normal-matter counterparts $M_N(u)$ and $f_N(u)$, respectively,

$$y(u) = h_c \rho_p^{cc} + 4\pi M_N(u)/h_c, \qquad (12)$$

$$f(u) = f_N(u)/h_c. \qquad (13)$$

This then allows for smooth matching of the right-hand sides of the two governing equations (6) and (11).

Numerics.—The governing equation for superconducting equilibria (11) is harder to solve than the Grad-Shafranov equation (6). The latter is itself unusual in having the argument u appear on both the left- and right-hand sides, but the superconducting version also has the quantity Π, which implicitly involves derivatives of u,

$$\Pi = \frac{|\nabla u|}{\sqrt{\varpi^2 \rho_p^2 - f^2}}. \qquad (14)$$

With a direct solution seeming infeasible, this problem is suited to a numerical iterative method, where both the left- and right-hand sides are gradually updated until the scheme converges and produces a consistent solution for u. We use an adapted self-consistent field method [26,30], allowing for high multipolar structure, and employ an underrelaxation step for the solution of Eq. (11).

Using the virial theorem [31], we confirm that our results are indeed equilibrium solutions, with a relative error of the order of 10^{-5}. This error decreases accordingly with increasing resolution. As another check, we have constructed equilibria with increasing field strength (and hence distortion), confirming that the induced distortion scales in the expected manner: linearly in $H_{c1}B$ [9,32]. Details of these tests and the numerical method will be presented in a later paper.

Results.—We consider a stratified stellar model with a composition gradient; i.e., the proton fraction ρ_p/ρ varies within the star. Various forms for the magnetic functions are permissible. We choose $f_N(u) = a(u - u_{\rm int})^\zeta$ when $u > u_{\rm int}$ and $f_N(u) = 0$ otherwise, where $u_{\rm int}$ is the largest contour of u (i.e., field line) that closes within the star; this avoids having an exterior current. We also take $M_N(u) = \kappa u^2$ unless otherwise stated. Here a, ζ, and κ are constants related to the strength of the magnetic field components. The figures are presented in dimensionless units, since the structure of the magnetic field is essentially independent of its strength.

We begin by looking at a purely poloidal field, in Fig. 1. This is broadly similar to the corresponding field in a normally conducting star, with the field strength attaining a maximum $B_{\rm max}$ deep in the star and vanishing in the center of the closed field-line region. The main difference is that the superconducting star shown has a far larger weak-field region than a model with normal protons; more specifically, a larger volume of the superconducting star has a field strength $B < 0.1B_{\rm max}$. This effect is even more pronounced for models with $M_N(u) = \kappa u$.

In Fig. 2, we show a typical mixed poloidal-toroidal field configuration. For stars with normal protons the toroidal

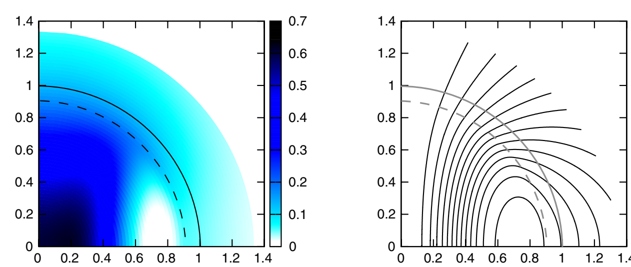

FIG. 1 (color online). Structure of a poloidal magnetic field in a superconducting star. On the left we show the magnitude of the field and on the right its direction (i.e., the field lines). The stellar surface R_* is indicated with the solid arc at a dimensionless radius of unity, while the dashed line at $0.9R_*$ shows the location of the crust-core boundary.

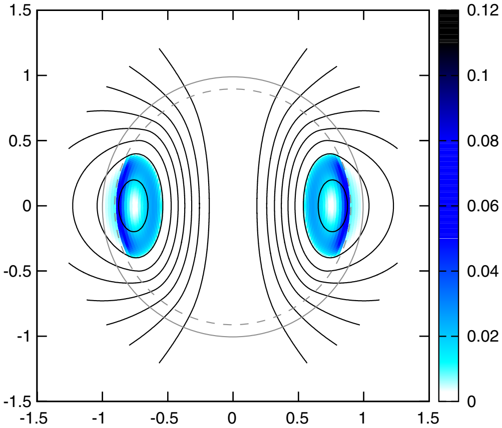

FIG. 2 (color online). A mixed poloidal-toroidal magnetic field in a superconducting star. We show the poloidal field lines in black and the magnitude of the toroidal-field component with the gray (color) scale. The stellar surface and crust-core boundary are indicated with the solid and dashed circles, respectively. Note that the toroidal component has a tubular structure, vanishing in the center of the closed field-line region.

component fills the weak-field region shown in Fig. 1, producing a twisted-torus configuration [26]. This is partially true here, too—but at the center of the closed field-line region, where the poloidal field vanishes, the toroidal component vanishes, too. The resulting toroidal-field geometry is hence tubular (in three dimensions). The mixed-field configuration in Fig. 2, like all those we have found, is dominated by the poloidal component; the toroidal component contributes only 0.7% of the total magnetic energy. This may be related to the fact that Eq. (11) has a purely poloidal limit but no purely toroidal one.

Next we look at the magnetically induced ellipticity $\epsilon = (Q_{\rm eq} - Q_{\rm pole})/Q_{\rm eq}$, where $Q_{\rm eq}$ and $Q_{\rm pole}$ represent the components of the star's quadrupole moment along the equator and pole, respectively. We rescale to a typical 1.4-solar mass neutron star with a radius of 10 km and assume a purely poloidal field. The mass of the normal-fluid crust is small, so the ellipticity scaling is well approximated by that of a purely superconducting star,

$$\epsilon = 3.4 \times 10^{-8}\left(\frac{B_s}{10^{12}\ {\rm G}}\right)\left(\frac{H_{c1}(0)}{10^{16}\ {\rm G}}\right), \qquad (15)$$

where B_s denotes the surface field strength at the pole. We adopt a central critical field of $H_{c1}(0) = 10^{16}$ G, using the approximate formula given in Ref. [11]. For comparison, in the same stratified two-fluid model but with normal protons we find (using the code described in Ref. [26])

$$\epsilon = 2.6 \times 10^{-11}\left(\frac{B_s}{10^{12}\ {\rm G}}\right)^2. \qquad (16)$$

A simple way to approximate the ellipticity of a superconducting star is to take a result for normal matter and scale it up by a factor H_{c1}/B, taking $H_{c1} = 10^{15}$ G.

TABLE I. The ratio $\bar{B}/B_s$ of average internal field to polar field and the prefactor k_ϵ of the ellipticity formula (15), for different poloidal-field configurations [specifically, different relations between the stream function u and the magnetic-force function $M_N(u)$].

$M_N(u)$ scaling	u	u^2	u^3	u^4
$\bar{B}/B_s$	1.5	2.1	2.7	2.5
$k_\epsilon[10^{-8}]$	2.5	3.4	3.9	3.5

Comparing our ellipticity formulas (15) and (16), both from self-consistent calculations, we see that for a given B_s this approach would underestimate the star's distortion by around 30%. Choosing $M_N(u)$ as a higher power of u increases the ratio of average internal field $\bar{B}$ to B_s and hence the ellipticity at a fixed value of B_s. This is summarized in Table I.

Discussion.—We have described a method to solve for the magnetic field in a neutron star with type-II superconducting protons. The magnetic force is more complicated in this case than for normal matter, but for an axisymmetric equilibrium we show that it may be simplified to a single differential equation in the stream function and the local field strength, in analogy with the Grad-Shafranov equation of normal matter. We solve this by using an iterative scheme, presenting the first self-consistent models of a superconducting neutron star (other than the special case of a purely toroidal field).

Perhaps the most notable difference from normal-matter models is the generic appearance of a region in the star where the field strength vanishes. This may have repercussions for the dynamics of a neutron star; in normal matter, it is known that such a region leads to an instability [33,34]. Whether such an instability occurs in superconducting matter is an interesting open question.

The interior field of a neutron star cannot necessarily be inferred from the observed exterior dipole field. If many poloidal field lines close within the star, or if there is a strong toroidal component, the internal field could be much stronger than expected from outside—a "hidden" energy reservoir for the star. In our models the average interior field strength is 1.5–2.7 times that at the polar surface. The contribution of the toroidal component appears to be generically small, however; in our equilibria it accounts for less than ~1% of the magnetic energy. This is different from recent simulations for main-sequence stars, which found stable equilibria with *large* toroidal components [35].

A leading theory for the origin of neutron star magnetism (for magnetars, in particular) is that dynamo action in the young star generates a strong, dominantly toroidal field [36]—in contrast to our equilibria for mature neutron stars. This may cast doubt on the validity of our equilibria or the dynamo scenario. Alternatively, both could be reasonable—then, as a hot young neutron star cools to a multifluid state with superconducting protons, its strong toroidal field would no longer be in equilibrium. It would have to undergo large-scale rearrangement to a poloidal-dominated equilibrium, a potentially violent transition which could be observable.

The effect of superconductivity on neutron star magnetic fields has been, to date, neglected by the vast majority of studies. This Letter demonstrates that in the simplest equilibrium models it may be accounted for by using similar techniques as for normal-matter stars. Two key issues which future work should address are the presence of a magnetic force on the neutrons and the physics at the crust-core boundary. Beyond this, superconductivity will surely also play an important role in the evolution and dynamics of neutron stars.

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