Many-Body Localization in One Dimension as a Dynamical Renormalization Group Fixed Point

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We formulate a dynamical real space renormalization group (RG) approach to describe the time evolution of a random spin-1/2 chain, or interacting fermions, initialized in a state with fixed particle positions. Within this approach we identify a many-body localized state of the chain as a dynamical infinite randomness fixed point. Near this fixed point our method becomes asymptotically exact, allowing analytic calculation of time dependent quantities. In particular, we explain the striking universal features in the growth of the entanglement seen in recent numerical simulations: unbounded logarithmic growth delayed by a time inversely proportional to the interaction strength. This is in striking contrast to the much slower entropy growth as loglog$t$ found for noninteracting fermions with bond disorder. Nonetheless, even the interacting system does not thermalize in the long time limit. We attribute this to an infinite set of approximate integrals of motion revealed in the course of the RG flow, which become asymptotically exact conservation laws at the fixed point. Hence we identify the many-body localized state with an emergent generalized Gibbs ensemble.

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What is the effect of interactions on Anderson localization? One common wisdom is that any amount of interaction will give rise to collective excitations that could assist transport at nonvanishing temperature even if single particle states are all localized. But the belief that there are no strict many-body insulators at $T > 0$ has been challenged by theoretical arguments, dating as far back as Anderson's original paper, which suggest a many-body localization transition marking a critical point in the transport properties of a closed quantum system [1,2]. The idea has recently gained support from numerical studies [3–6]. Furthermore, simulations of one-dimensional systems have revealed remarkably universal behavior of the dynamics in the putative many-body localized state [7–9]. For example, the time evolution following a quench shows unbounded logarithmic growth of the entanglement entropy, in striking contrast to the bounded growth seen in a system of noninteracting fermions.

In this Letter we provide a theory of the many-body localized state using a novel renormalization group (RG) to describe the dynamics of one-dimensional random systems. For certain initial conditions we can establish a many-body localized state as an infinite randomness fixed point of the dynamics near which the RG scheme is asymptotically exact. Our results explain many of the universal features found in the numerical simulations mentioned above.

As a starting point we consider the Hamiltonian of the random spin-1/2 XXZ chain without local Zeeman fields:

$$H = \sum_i \frac{J_i}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ + 2\Delta_i S_i^z S_{i+1}^z). \quad (1)$$

The couplings J_i and anisotropy parameters Δ_i on sites i are random variables drawn from uncorrelated probability distributions. The couplings may be positive or negative and we assume $|\Delta_i| < 1$. The spins in (1) can be mapped using a Jordan-Wigner transformation to spinless fermions with nearest-neighbor interactions, subject to bond disorder.

To study the propagation of information through the chain, we investigate the time evolution of the system starting from a nonentangled initial state, which for simplicity we take as an antiferromagnetic Néel state with spins pointing along the z axis. We shall see that this choice of initial state greatly simplifies the scheme and allows us to obtain well-controlled results for the dynamics at long times.

RG scheme.—Our approach utilizes the local separation of scales induced by strong disorder to gradually eliminate degrees of freedom oscillating at high frequencies. Thus the role played by randomness is similar to that in the standard RG scheme used to find ground state correlations of disordered spin chains [10–12]. However, instead of focusing on the ground state of the system, our scheme targets the evolution of the chain at long times starting from a specified (high energy) initial state. Such evolution is affected by the complete many-body energy spectrum. Similar ideas have been applied to solve classical dynamics in certain random [13,14] as well as clean [15] systems, but to our knowledge not to quantum dynamics.

Let Ω be the strongest exchange coupling on the chain. The dynamics at the shortest time scales are oscillations of frequency Ω between the $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ of the pair of spins on the strong bond. On these time scales the strong pair is effectively decoupled from its typically slower neighbors.

To capture the dynamics on time scales much longer than Ω^{-1}, we perturbatively eliminate the rapid oscillations, thus renormalizing the couplings between the remaining slow degrees of freedom.

Working in the interaction picture with respect to the Hamiltonian of the strong pair, we compute the evolution of the density matrix $\rho(t) = U_I^\dagger(t)|\psi_0\rangle\langle\psi_0|U_I(t)$ to second order in the coupling to the rest of the chain. The ensuing time dependence, averaged over the rapid oscillations, can be matched term by term to that generated by an effective Hamiltonian $\rho(t) = \exp(-iH_{\rm eff}t)|\psi_0\rangle\langle\psi_0|\exp(iH_{\rm eff}t)$, which describes the slow dynamics at time scales $t \gg \Omega^{-1}$. The scheme is controlled by the ratio of the typical coupling to the neighboring spins over the coupling on the strong bond. If, as the scheme is iterated, the distribution of coupling constants broadens, then the RG becomes increasingly well controlled, or even asymptotically exact if the system flows to infinite randomness [12].

The effective Hamiltonian is particularly simple for the chosen initial conditions. At second order of perturbation theory the $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ are never populated and therefore truncated from the Hilbert space. The retained states $|\pm\rangle = 2^{-1/2}(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle)$ of the strong pair can be taken as the $\uparrow/\downarrow$ states of a new pseudospin variable $\vec{S}_n$, which initially points along the positive or negative x axis. The effective Hamiltonian resulting from the procedure can then be written as

$$H_{\rm eff} = H_{\rm chain} + h_n S_n^z + \frac{J_L J_R}{2\Omega(1-\Delta_S^2)}(S_L^+ S_R^- + S_L^- S_R^+) + \frac{\Delta_S J_L J_R}{2\Omega}\left[\frac{S_L^+ S_R^- + S_L^- S_R^+}{(1-\Delta_S^2)} - \frac{\Delta_L \Delta_R}{\Delta_S} S_L^z S_R^z\right] S_n^z, \quad (2)$$

where the L, R indices denote the left and right neighbors of the strong bond, Δ_S is the anisotropy parameter on the strong bond, h_n is an effective magnetic field that can be gauged away, and $H_{\rm chain}$ is the original Hamiltonian on the rest of the chain. Because $[H_{\rm eff}, S_n^z] = 0$, the time evolution can be computed separately for each eigenvalue $\pm\frac{1}{2}$ of S_n^z, using $H_{\rm eff}^\pm$ that does not depend on the operator $\vec{S}_n$. The different evolution under $H_{\rm eff}^\pm$ together with the fact that the new spin starts in a superposition of $\uparrow$ and $\downarrow$ leads to entanglement between the effective spin on the strong bond and its two neighboring spins. Full entanglement is generated after a time $t_{\rm ent} = 2\Omega/(J_L J_R \Delta_S)$, set by the difference in the exchange constant in $H_{\rm eff}^\pm$. Note that entanglement is not generated in this process in the noninteracting system $\Delta = 0$.

Apart from generating entanglement, the difference between the evolution given $\uparrow_n$ or $\downarrow_n$ is not crucial for the subsequent dynamics in the sense that they both lead to the same recipe for renormalization of coupling constants. $H_{\rm eff}^+$ and $H_{\rm eff}^-$ have the same form as the original Hamiltonian, and we can directly read off the coupling generated between $\vec{S}_L$ and $\vec{S}_R$, neighboring the strong bond to the left and right, upon decimation of that bond: $\tilde{J} = J_L J_R/\Omega$ and $|\tilde{\Delta}| = |\Delta_L||\Delta_R|/4$, where we neglected the linear Δ correction to $\tilde{J}$. This approximation will be justified *a posteriori* by the fact that Δ flows to zero. The renormalization of the exchange coupling is then identical to that found in the random Heisenberg chain at $T = 0$ and leads to the random-singlet phase [10–12]. Note also that we keep only the absolute value of the anisotropy. The sign will randomize in the course of the RG flow because it depends on the state of S_n^z.

The RG steps are iterated to produce a flow of the probability distributions with decreasing cutoff Ω starting from the microscopic cutoff Ω_0. Using the scaling variables $\zeta = \ln\frac{\Omega}{J}$ and $\beta = -\ln|\Delta|$, and $\Gamma = \ln(\Omega_0/\Omega) = \ln(\Omega_0 t)$, we obtain the following equation for the joint probability distribution $P(\zeta, \beta; \Gamma)$:

$$\frac{\partial P}{\partial \Gamma} = \frac{\partial P}{\partial \zeta} + \rho(0;\Gamma)\int_0^\infty d\beta_L d\beta_R d\zeta_L d\zeta_R \delta(\zeta - \zeta_L - \zeta_R) \times \delta(\beta - \beta_L - \beta_R - \ln 4) P(\zeta_L, \beta_L; \Gamma) P(\zeta_R, \beta_R; \Gamma), \quad (3)$$

where $\rho(\zeta;\Gamma) = \int d\beta P(\zeta, \beta; \Gamma)$ is the distribution of ζ. Note that even if initially the variables ζ and β are independent, a correlation builds up in the course of renormalization in the same way as it is generated in the ground state [12].

By integrating over β we obtain an equation for $\rho(\zeta;\Gamma)$,

$$\frac{\partial \rho}{\partial \Gamma} = \frac{\partial \rho}{\partial \zeta} + \rho(0;\Gamma)\int_0^\infty d\zeta_L d\zeta_R \delta(\zeta - \zeta_L - \zeta_R)\rho(\zeta_L)\rho(\zeta_R). \quad (4)$$

This equation is identical to the flow leading to the random-singlet ground state and it is solved by the same ansatz [12], $\rho(\zeta;\Gamma) = a(\Gamma)e^{-a(\Gamma)\zeta}$ with $a(\Gamma) = (\Gamma + 1/a_0)^{-1}$.

Of course the above solution includes only partial information on the fixed point of the dynamics. As in Ref. [12], important information for calculation of physical quantities is held in the conditional average of the interaction variable β given a value of ζ on the same bond, $\bar{\beta}(\zeta, \Gamma) \equiv \int_0^\infty d\beta \beta P(\zeta, \beta; \Gamma)/\rho(\zeta;\Gamma)$. We derive the equation for this moment by multiplying Eq. (3) by β and then integrating over β:

$$\partial_\Gamma \ln(\bar{\beta}(\zeta)\rho) = \partial_\zeta \ln(\bar{\beta}(\zeta)\rho) + \frac{2a(\Gamma)}{\bar{\beta}(\zeta)}\int_0^\zeta d\zeta' \bar{\beta}(\zeta'). \quad (5)$$

Neglecting the ln4 in (3) is justified near the fixed point since the typical β flows to ∞. Substituting the solution for $\rho(\zeta, \Gamma)$ in Eq. (5), we find the solution

$$\bar{\beta}(\zeta) = \frac{1}{b_0}(a_0\Gamma + 1)^\phi\left(1 + \frac{\zeta\phi}{\Gamma + 1/a_0}\right), \quad (6)$$

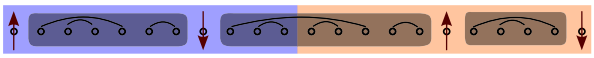

FIG. 1 (color online). Schematic illustration of remaining spins and clusters of decimated pairs in the renormalized chain at time t.

where $\phi = (1+\sqrt{5})/2$ is the golden ratio and b_0 is determined by the initial condition.

An important ingredient for calculation of physical properties is the distance between remaining spins, or length of decimated clusters, at time t (see Fig. 1). Since the flow (4) is formally the same as in Ref. [12], we similarly obtain $L_\Gamma = (a_0\Gamma+1)^2 = [a_0 \ln(\Omega_0 t)+1]^2$, which behaves as $\ln^2(\Omega_0 t)$ at long times.

Results.—First, as an immediate corollary of the relation $L(t)$, we can obtain the decay of the Néel order parameter. This is given by the fraction of undecimated spins, still frozen in a Néel order at time t: $m_s = 1/L(t) = 1/[a_0 \ln(\Omega_0 t)+1]^2$. It is interesting to contrast this behavior with the decay of the staggered moment in the analogous quench of a clean XXZ model, found to be oscillatory (for $\Delta < 1$), with an envelope that decays exponentially in time [16].

Next, to gain information on particle transport and thermalization, we compute the growth of the total particle number fluctuation and of the entanglement entropy in a subsystem consisting of half the chain. Each decimated pair has a conserved integer particle number in the RG scheme. Therefore, only decimated pairs that cut the interface between the two half chains contribute to the particle number fluctuation in the subsystem. Such oscillating pairs add $1/8$ to the number fluctuation on time average. Computing the total particle number fluctuation then amounts to counting the number of decimated bonds that cut the interface [17]: $N_p \approx \int^\Gamma d\Gamma' a(\Gamma') = \ln(\Gamma + 1/a_0)$. Hence, the particle number fluctuation grows extremely slowly as $\langle \delta N^2 \rangle = (1/24)\ln[\ln(\Omega_0 t)]$ at long times. Interestingly, this result is independent of the interaction strength Δ.

On the other hand, we shall see that the interaction has a dramatic effect on the growth of the entanglement entropy between the two halves. In the noninteracting system ($\Delta_i = 0$), the second line of (2) is zero and hence no entanglement is generated between a decimated pair and the rest of the chain. As for the particle number fluctuation, the only source of entanglement then is decimated pairs whose spins reside on opposite sides of the interface. During an oscillation period, such a pair contributes a time average of $S_p = 2 - 1/\ln 2 \approx 0.557$. The growth of the entropy is then similar to that of the particle number fluctuation:

$$S_0(t) \approx S_p \frac{1}{3} \ln[\ln(\Omega_0 t) + 1/a_0]. \tag{7}$$

We can generalize this result (for $\Delta_i = 0$) to a quench from an arbitrary Ising state with a fraction q of antialigned neighbors. Because q is an invariant of the RG and aligned pairs do not contribute to the entropy, the prefactor in (7) changes to qS_p.

Interactions lead to a new source of entanglement. A pair decimated at time t_1 will eventually get entangled with the neighboring spins according to Eq. (2) after a characteristic time $t_{\rm ent}(t_1) = 2\Omega_1/(J_1^2\Delta_1)$. In particular, from $t = 0$ entanglement will be generated by interactions only after a delay time $t_{\rm delay} = 2\Omega_0/(J_0^2\Delta_0) = 2(\Omega_0/J_0)(1/J_0^z)$, where $J_0^z \equiv J_0\Delta_0$ is the typical value of the bare interaction energy.

The interaction-generated entanglement entropy found at time t originates from entanglement of pairs eliminated at an earlier time $t_1 = t - t_{\rm ent}$ or $\Gamma_1 = \ln\Omega_0 t_1$. To estimate this contribution to the entropy, we recall that spins on the renormalized chain at time t_1 are separated by clusters of length $L(\Gamma_1)$ of decimated spins oscillating at higher frequencies. By the time t that a pair of spins decimated at t_1 entangles with their neighbors, the pseudospins inside the decimated clusters must also be entangled with each other. Hence, by the observation time t entanglement had propagated to a distance $L(\Gamma_1)$ giving rise to entanglement entropy $S \approx 0.5L(\Gamma_1) \approx 0.5(a_0\Gamma_1+1)^2$. The factor 0.5 stems from the number of available degrees of freedom: the two states with aligned spins in each decimated pair remain unpopulated and therefore do not contribute to the entropy. To write this as a function of the time t, we use the relation between t and t_1:

$$t = t_1 + t_{\rm ent} = t_1\left(1 + \frac{2\Omega_1^2}{J_L J_R \Delta_1}\right) \approx t_1 \frac{2\Omega_1^2}{J_L J_R \Delta_1}. \tag{8}$$

We now take the logarithm of both sides and replace the scaling variables by their appropriate average values $\zeta \to 1/a(\Gamma_1)$ and $\beta_1 \to \bar{\beta}(\zeta = 0; \Gamma_1)$. Note the importance of correlations: we needed the average of β on the bonds with strongest J ($\zeta = 0$) rather than the global average of β.

Using the solutions for $a(\Gamma)$ below (4) and (6), for the typical values, we find $\Gamma = 3\Gamma_1 + \frac{1}{b_0}(a_0\Gamma_1+1)^\phi + 2/a_0 + \ln 2$. By inverting this equation to obtain $\Gamma_1(\Gamma)$, we can find $S(\Gamma) = 0.5L(\Gamma_1(\Gamma))$. In limiting regimes the equation can be inverted analytically. At long times, when the term Γ_1^ϕ dominates the right-hand side, we have $a_0\Gamma_1 = [b_0(\Gamma - 2/a_0 - \ln 2)]^{1/\phi} - 1$, while at short times, when the linear term dominates, we have $\Gamma_1 = \frac{1}{3}(\Gamma - \frac{2}{a_0} - \frac{1}{b_0} - \ln 2)$.

The crossover time t^* separating the two regimes depends on the initial conditions through the coefficients of the terms Γ_1 and Γ_1^ϕ. If $b_0 \gg a_0$, that is for stronger disorder in hopping than in the interactions, we have $t^* = t_{\rm delay}\exp[6(3b_0/a_0)^\phi/a_0]$. In the opposite regime $b_0 \ll a_0$, the term Γ_1^ϕ dominates from the outset and $t^* = t_{\rm delay}$. We can now write an expression for the growth of the entanglement entropy valid in the limiting regimes:

$$S(t) \approx \frac{1}{2}\left(\frac{\ln(t/t_{\rm delay})}{\ln(\Omega_0/J_0)}+1\right)^2\theta(t-t_{\rm delay})\theta(t^*-t) + \frac{1}{2}\left(\frac{\ln(t/t_{\rm delay})}{\ln(1/\Delta_0)}+1\right)^{2/\phi}\theta(t-t^*)-\frac{1}{2}. \quad (9)$$

Interestingly, Eq. (9) gives the unbounded logarithmic growth of the entanglement entropy seen in the numerical simulations, and even the delay of this interaction induced growth by a time that scales as the inverse interaction strength [9]. Also consistent with these simulations is the much slower ($\sim \ln\ln t$) increase of the particle number fluctuation found in the RG approach. Indeed, the particle number fluctuations are in general only a lower bound of the entanglement entropy [18]. One cannot, however, make a detailed comparison with the numerics done for a somewhat different model including on-site Zeeman field disorder (but see the Supplemental Material [19]).

Having found that the entanglement entropy increases without bound, it is natural to ask if this leads to thermalization. To address this issue let us consider the saturation of the entanglement entropy in a finite subsystem of length L_s. Equation (9) implies that the entropy will approach its maximal value S_∞ after a time $t_{\rm sat} \approx t_{\rm delay}\exp[-\ln(\Delta_0)L_s^{\phi/2}]$. Does the saturation value S_∞ correspond to a state in thermal equilibrium?

Provided we start from a symmetric distribution of Δ_i such that $\langle\Delta_i\rangle = 0$, then the initial Néel state has zero mean energy, exactly in the middle of the many-body energy spectrum. If this state thermalized following the quench, the entanglement entropy would have to saturate to its infinite temperature value of L. But as we have pointed out above, the RG flow implies a saturation entropy that is at most half of the infinite temperature value because half of the degrees of freedom remain frozen in the dynamics.

We can rephrase this fact more precisely in terms of local conservation laws. For each decimated pair we identify an integral of motion $I_p = (S_1^z S_2^z)_p$. Conservation of I_p reflects the fact that within the perturbative RG scheme a pair of decimated spins never flip their relative orientation. In the case of an initial Néel state we also have a local particle conservation on each oscillating pair, i.e., $(S_1^z + S_2^z)_p$. Although these are approximate conservation laws, they become asymptotically exact near the infinite randomness fixed point. Given any long observation time t, there are always an infinite number of quantities that are conserved for a time longer than t. It is plausible to conjecture that there are in fact exact integrals of motion related to the approximate local ones up to exponentially small long-range tales. We conclude that the long time steady state of the chain with nonvanishing interaction is characterized by the generalized Gibbs ensemble (GGE) [20], which describes thermalization within a subspace constrained by the values of the emergent integrals of motion I_p.

It is important to note that the long time steady state attained by the noninteracting system $\Delta_i = 0$ is markedly different. The extremely slow increase of the entanglement entropy as $\ln\ln t$ given by Eq. (7) together with the relation between length and time scales $\ln\Omega_0 t = \Gamma \sim \sqrt{L}$ imply saturation of the entanglement entropy to $S_\infty \approx \frac{S_p}{6}\ln L$. This result, as well as the $\ln\ln t$ growth of the entropy, matches with numerical results obtained for the random transverse field Ising chain [21] that can be similarly described by a model of free fermions.

Conclusion.—Using a real space RG scheme formulated in real time, we gave a dynamical description of a many-body localized state in a random spin chain, equivalent to interacting fermions with random hopping. Within this approach the localized state is characterized by a flow to an infinite randomness fixed point. Solution of the flow equations allows us to characterize this state in a rather detailed way. The results are consistent with and explain the universal features found in recent numerical simulations done on a similar, albeit not identical, model [9].

Particle localization is manifest in the extremely slow growth $\sim \ln\ln t$ of the particle number fluctuations in half the system that is seen in both the interacting and noninteracting systems. The entanglement entropy S reveals a dramatic difference between the Anderson localized state of noninteracting fermions and the many-body localized state established with interactions. In the noninteracting system S grows together with the particle number fluctuation as $S(t) \sim \log\log t$ and saturates to a nonextensive value $\sim \ln L$ in a finite system. Interactions lead to much faster growth of $S(t)$ as $\log^{2/\phi} t$ at long times, but they take effect only after a delay time that scales as the inverse of the interaction strength $t_{\rm delay} \sim 1/J^z$. Furthermore, the $\log^{2/\phi} t$ behavior seen in the long time limit is preceded by $\log t$ growth up to an intermediate time scale $t_{\rm lin} \sim t_{\rm delay}(\Omega_0/J_0)^2 \gg t_{\rm delay}$. It is interesting to note that the growth of entanglement as $\ln^{2/\phi} t$ exceeds the upper bound $\sim \ln t$ proved for noninteracting Anderson localized chains [22].

The RG flow toward the infinite randomness fixed point has direct consequences on the equilibration in this system. In a subsystem of length L the entanglement entropy saturates to an extensive value $S_\infty \sim L$, which is, however, smaller than it would reach had the system attained true thermal equilibrium. We attribute the lack of thermalization to an infinite set of emergent integrals of motion, which become asymptotically exact conservation laws near the infinite randomness dynamical fixed point. The dynamics of the system can therefore be viewed as thermalization within a GGE characterized by the emergent set of conserved quantities, a possibility suggested in Refs. [23,24], Here we demonstrated that such a GGE emerges in a nonintegrable random system as a dynamical fixed point of the renormalization group and captures the essence of a many-body localized state. The nature of the

critical point marking the transition to the normal thermalizing state remains an interesting question for future study as are generalizations of our scheme to more generic disorder models and initial states.

Before closing, we remark that the RG scheme does not account for resonances occurring between decimated pairs located far from each other on the chain. Processes that can resonantly change the values of the conserved operators I_p on the two pairs and thus lead to delocalization are potentially dangerous. However, a simple argument given in the Supplemental Material [19] shows that these are irrelevant near the infinite randomness fixed point. A more detailed analysis of resonances will be provided elsewhere [25].

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