Geometrical Properties of Turbulent Dispersion

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It is shown that the shape statistics of four correlated particles, known as a *tetrad*, in a direct numerical simulation of three-dimensional homogeneous isotropic turbulence agree very well with a simple diffusion equation with a separation-dependent eddy diffusivity. The latter is essentially an extension of Richardson's model for particle-pair dispersion to tetrads. It is also shown that the degree of elongation of the tetrad in the inertial subrange of a turbulentlike flow is controlled by the exponent, m, in an eddy diffusivity of the form $K(r) \propto r^m$ where r is the interparticle separation, becoming more elongated as m increases in the range $0 \leq m < 2$.

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Turbulence is well known to be very effective at mixing different fluids together. This can be seen in a casual observation of a power station chimney plume. What is also evident is the distortion of the plume as it travels downwind. A Lagrangian analysis of turbulence is well suited to the mathematical description of the plume. Its average spread is captured by Taylor's theory of one-particle diffusion [1] whereas the fluctuations in the plume's concentration are related to the dispersion of particle pairs, e.g., Ref. [2], the latter itself a celebrated problem in the study of turbulence [3]. Higher-order moments of the concentration field are related to the dispersion of clusters of particles. Here, we consider the statistics of four (correlated) particles, or *tetrads*. It will be shown that the distortion of tetrads in the inertial subrange in real turbulence, here taken to be a direct numerical simulation (DNS) of three-dimensional homogeneous isotropic turbulence, is very well approximated by a simple diffusion equation with a separation-dependent eddy diffusivity.

The form of the diffusivity is an extension of the model proposed by Richardson [3] to describe the dispersion of particle pairs to tetrads. The eddy diffusivity for the evolution of the position vector of the four particles, $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_4)$, is given by

$$K^{ij}_{\alpha\beta} = K_1 \delta^{ij} - \frac{1}{2} K^{ij}_{\Delta}(\boldsymbol{r}_{\alpha\beta}), \qquad \alpha, \beta = 1, \ldots, 4, \quad i, j = 1, 2, 3,$$

where K_1 is the constant diffusivity for independently moving particles at scales larger than the integral scale of turbulence, L, and K^{ij}_{Δ} is the eddy diffusivity for the separation $\boldsymbol{r}_{\alpha\beta} = \boldsymbol{x}_\beta - \boldsymbol{x}_\alpha$:

$$K^{ij}_{\Delta}(\boldsymbol{r}_{\alpha\beta}) = K_{NN}\delta^{ij} + (K_{LL} - K_{NN}) \frac{r^i_{\alpha\beta} r^j_{\alpha\beta}}{|\boldsymbol{r}_{\alpha\beta}|^2}.$$

The longitudinal part of K^{ij}_{Δ} is assumed to be

$$K_{LL}(r) = k_0 \frac{\varepsilon^{1/3} r^{4/3} L^{4/3}}{(r^2 + L^2)^{2/3}},$$

for some constant k_0, where ε is the mean kinetic energy dissipation rate. The transverse part, K_{NN}, can be derived by assuming that $K^{ij}_{\alpha\beta}$ satisfies the incompressibility condition. Note that $r_{\alpha\alpha} = 0$, i.e., $K^{ij}_{\alpha\beta}$ is then appropriate for a single particle. For $r \ll L$, $K_{LL} = k_0 \varepsilon^{1/3} r^{4/3}$, which is the standard form of the separation-dependent eddy diffusivity for particle pairs in the inertial subrange. For $r \gg L$, $K_{LL} = k_0 \varepsilon^{1/3} L^{4/3} = 2K_1$, and, hence, $K^{ij}_{\alpha\beta} = 0$ for $\alpha \neq \beta$ as is appropriate for uncorrelated particles that have separated beyond the integral scale. In practice, the diffusion equation is solved numerically via the equivalent stochastic differential equation. Henceforth, this model will be known as the Richardson model. It is similar to the Kraichnan white-noise model [4] which has been extensively studied for multiparticle dispersion, e.g., Ref. [5].

It is convenient to introduce a reduced set of coordinates that eliminates the center of mass and is orthogonal. Such coordinates are defined by, e.g., Ref. [6]

$$\boldsymbol{\rho}_1 = \frac{\boldsymbol{x}_2 - \boldsymbol{x}_1}{\sqrt{2}}, \qquad \boldsymbol{\rho}_2 = \frac{2\boldsymbol{x}_3 - \boldsymbol{x}_2 - \boldsymbol{x}_1}{\sqrt{6}},$$
$$\boldsymbol{\rho}_3 = \frac{3\boldsymbol{x}_4 - \boldsymbol{x}_3 - \boldsymbol{x}_2 - \boldsymbol{x}_1}{\sqrt{12}}.$$

These separation vectors can be embodied in the square matrix $\boldsymbol{P}$ whose columns are the three vectors $\boldsymbol{\rho}_\alpha$ and whose rows are the spatial coordinate. A moment of inertialike tensor can be defined $\boldsymbol{I} = \boldsymbol{P}\boldsymbol{P}^T$ whose eigenvalues are given by g_i ($i = 1, 2, 3$) [7]. The eigenvalues can be used to describe the shape and size of the tetrahedron. The squared volume, mean square area (over the four sides) and mean square separation (over the six sidelengths) of the tetrahedron are related to the three invariants of $\boldsymbol{I}$. The shape of the tetrahedron may be characterized in terms of $I_i = g_i / \sum_i g_i$. Ordering the eigenvalues so that

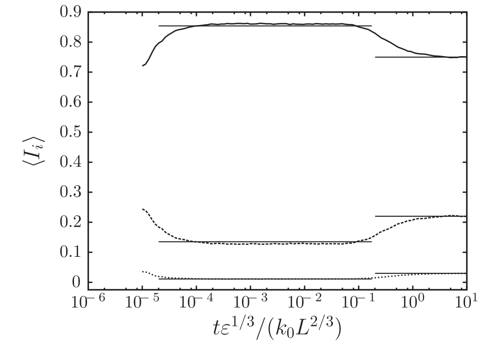

FIG. 1. Evolution of $\langle I_i \rangle$ ($i = 1, 2, 3$) for the Richardson model: $i = 1$ (solid line), $i = 2$ (dashed line) and $i = 3$ (dotted line). The horizontal lines are the values of $\langle I_i \rangle$ from the DNS of [8] ($\langle I_1 \rangle = 0.854$, $\langle I_2 \rangle = 0.135$, $\langle I_3 \rangle = 0.011$) and for independent Gaussian-distributed particles ($\langle I_1 \rangle = 0.75$, $\langle I_2 \rangle = 0.22$, $\langle I_3 \rangle = 0.03$; e.g., Refs. [7,8]).

$g_1 \geq g_2 \geq g_3$, an elongated tetrahedron has $I_1 \gg I_2, I_3$ and a flattened tetrahedron has $I_1, I_2 \gg I_3$.

Figure 1 shows the evolution of $\langle I_i \rangle$, where $\langle \cdot \rangle$ indicates an ensemble average, for the Richardson model. Also indicated are the inertial subrange values of $\langle I_i \rangle$ for the DNS data of Ref. [8] which has a Taylor-scale Reynolds number, $R_\lambda \approx 284$. (Note that the tetrads that contribute to these statistics were filtered to ensure that they lie fully within the inertial subrange [8].) It is clear that the Richardson model and DNS data agree very well.

A comparison of the probability density functions (PDFs) of I_i provides a more stringent test of the Richardson model. Figure 2 shows the joint PDFs of, respectively, (I_1, I_2) and (I_2, I_3) for the DNS, Richardson model, and independently moving particles. Also shown in Fig. 2 is the triangle of admissible values of I_1, I_2, and I_3 the sides of which are the limiting shapes for a tetrahedron: $I_1 + 2I_2 = 1$ (equivalently $I_2 = I_3$), $I_1 + I_2 = 1$ (equivalently $I_3 = 0$) and $I_1 = I_2$ (equivalently $2I_2 + I_3 = 1$) [9]. Figure 2 shows good agreement between the Richardson model and the DNS data and indicates that in the inertial subrange the tetrads are most likely to be elongated which, as has been noted before, e.g., Refs. [8,9] is distinct from the preferred shape of independently moving particles which tends to be flatter.

The good agreement between the DNS and the Richardson model for the shape statistics indicates that knowledge of the two-particle diffusivities in real turbulence is sufficient for determining the shape statistics and that nondiffusive effects do not play a significant role. Moreover, the agreement shown in Figs. 1 and 2 is better than has been observed for particle-pair exit-time PDFs [10,11] and much better than for particle-pair separation statistics, particularly the elusive t^3-law, e.g., Refs. [10,12],

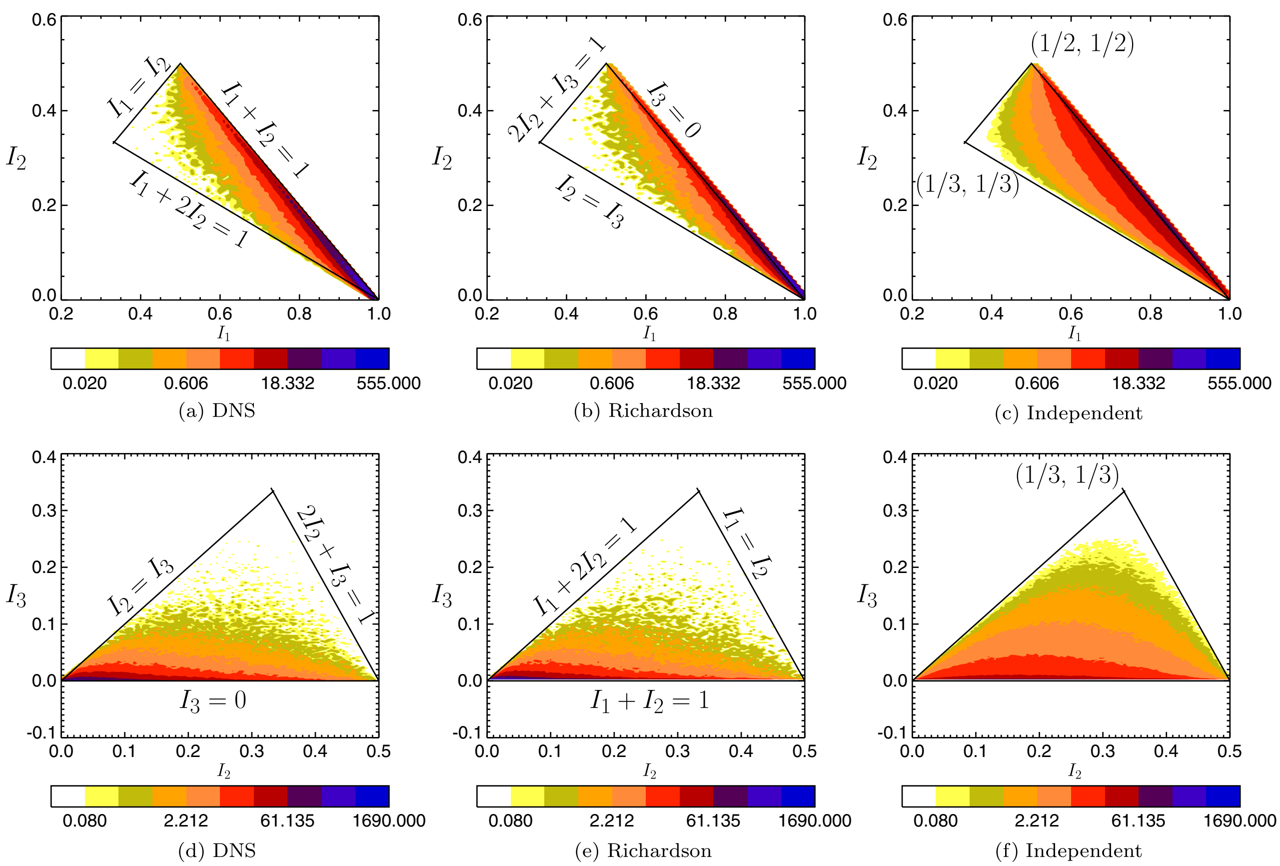

FIG. 2 (color online). Joint PDFs of (I_1, I_2) (a)–(c) and (I_2, I_3) (d)–(f). The contours are logarithmically spaced.

where t is the travel time. The reasons for this are not clear but are indicative of a more rapid decorrelation of the tetrads' shape (compared with their size) and the consequent loss of memory of the initial configuration; the size statistics appear to be more affected by finite R_λ-effects. The preference for elongated shapes in the inertial subrange occurs for any flow field in which the dispersion is reasonably well approximated by a (longitudinal or transverse) diffusivity of the form $K(r) \propto r^{4/3}$. Indeed, for a more generalized Richardson model in which $K(r) \propto r^m$, the "degree of elongation" varies with m. Figure 3 shows that the degree of elongation increases with increasing m. As $m \to 2$, the time taken for any two initially close particles to separate to $O(L)$ tends to infinity. If one particle is perturbed in some way in this limit, it will continue to move away from the other particles such that $I_1 \to 1$ and $I_2, I_3 \to 0$.

An example of a turbulentlike flow in which the dispersion is characterized by a separation-dependent diffusivity whose exponent is not equal to $4/3$ is kinematic simulation (KS). KS is a synthetic turbulent flow with a prescribed energy spectrum formed by the linear superposition of random Fourier modes (e.g., Ref. [13]). There is only a notional energy dissipation rate and L and η, where η is the Kolmogorov scale, are prescribed. The wave numbers are logarithmically spaced so that the energy spectrum is less frequently sampled than is the case in DNS making it a computationally efficient method of generating a wide range of scales. KS has been extensively used to study particle-pair dispersion, e.g., Ref. [14] and the dispersion of triangles and tetrads [15]. However, the lack of coupling between the modes means that there is no sweeping of the small scales by the large scales (as is the case in real turbulence). A consequence of this is that a pair of particles is swept through the small eddies by the large-scale sweeping velocity. This is more easily understood when a large mean velocity, U, is added to the flow to exaggerate the lack of sweeping. Thomson and Devenish [16] showed that in KS the eddy decorrelation time scale is of order r/U, the time taken for a pair of particles to be swept through an eddy of order r by the large-scale sweeping velocity. This time scale is much smaller than the classical inertial subrange time scale, $r^{2/3}/\varepsilon^{1/3}$. The appropriate form of the eddy diffusivity in KS with a large mean flow is then $K(r) \sim \varepsilon^{2/3} r^{5/3}/U$ [16]. The lack of sweeping means that the physics of the separation process in KS is different from that in real turbulence and this is reflected in the different scaling. The simulation conducted for this study used a Kolmogorov energy spectrum, 1200 Fourier modes, an inertial subrange of six decades ($\eta/L = 10^{-6}$) and an initial interparticle separation of $r_0/L = 10^{-5}$ (details of the construction of the velocity field in KS can be found in, e.g., Ref. [16]). Even for such a large range of scales, the tetrads exhibit the predicted inertial-subrange scaling for no more than a couple of decades in time; this is because here the particles' separation grows like t^6 [16]. Figure 4 shows the evolution of $\langle I_i \rangle$ computed from a 3D KS with $U = 10\sigma_u$, where σ_u is the velocity standard deviation. It is clear that kinematic simulation agrees well with the Richardson model with $m = 5/3$.

The shape statistics of real and synthetic turbulent flows in the inertial subrange are well approximated by a generalized Richardson model for which the limiting values of $\langle I_i \rangle$ are not equal to one or zero. That such a result is not a more general property of nonlinear flows can be easily demonstrated by considering the dispersion of tetrads in a smooth chaotic flow. The Lorenz system [17] is taken as an example of a chaotic flow:

$$\frac{dx}{dt} = u(x, y, z) = -\sigma(x - y),$$
$$\frac{dy}{dt} = v(x, y, z) = rx - y - xz,$$
$$\frac{dz}{dt} = w(x, y, z) = -bz + xy,$$

where $\sigma = 10$, $r = 28$, and $b = 8/3$. A regular tetrahedron with interparticle separation much less than the size of the attractor is constructed from four neighboring initial

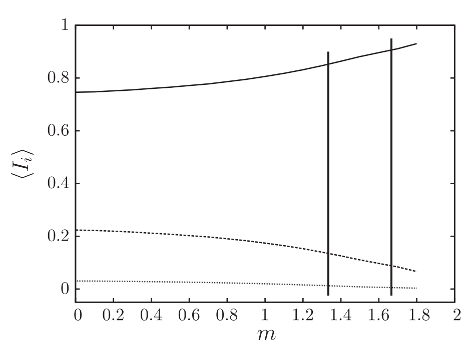

FIG. 3. Variation of $\langle I_i \rangle$ ($i = 1, 2, 3$) for the Richardson model with m: $i = 1$ (solid line), $i = 2$ (dashed line) and $i = 3$ (dotted line). The values of $m = 4/3$ and $m = 5/3$ are marked by vertical black lines.

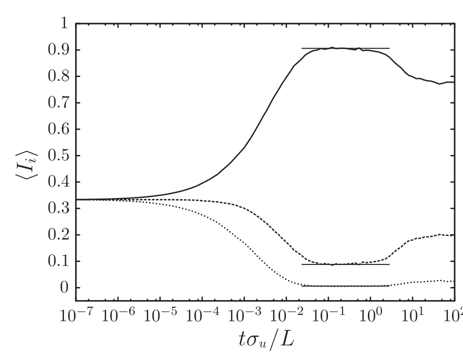

FIG. 4. As Fig. 1, but for kinematic simulation. The horizontal lines are the values of $\langle I_i \rangle$ for the Richardson model with $m = 5/3$: $\langle I_1 \rangle = 0.906$, $\langle I_2 \rangle = 0.088$, $\langle I_3 \rangle = 0.006$.

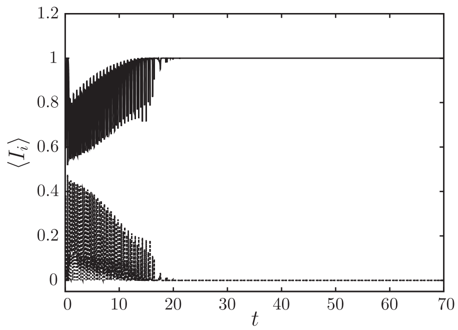

FIG. 5. As Fig. 1, but for Lorenz flow.

points. The evolution of $\langle I_i \rangle$ is shown in Fig. 5. After an initial transient regime the limiting values $\langle I_1 \rangle = 1$ and $\langle I_2 \rangle = \langle I_3 \rangle = 0$ are reached. This result is, of course, expected for a smooth chaotic flow and can be readily understood in terms of the Lyapunov exponents of the flow which for the Lorenz model are $\lambda_1 = 0.9$, $\lambda_2 = 0$, and $\lambda_3 = -14.57$, e.g., Ref. [18]. Since $g_i \sim \exp(\lambda_i t)$, it follows that $I_1 = 1$ and $I_2 = I_3 = 0$ at large times.

It has been demonstrated that the distortion of tetrads in the inertial subrange of a DNS of homogeneous isotropic turbulence with $R_\lambda \approx 284$ is well approximated by a Richardson-type diffusion equation with an appropriate diffusivity and, more generally, that the degree of elongation depends on the value of the exponent, m, in $K(r) \propto r^m$. The latter was illustrated using kinematic simulation, a turbulentlike flow field in which the physics of the tetrads' distortion differs from that in real turbulence. The agreement between the DNS shape statistics and those of the Richardson model is significantly better than the agreement between DNS and the Richardson model that has been seen elsewhere in other statistics such as the particle-pair separation (which may be due to finite R_λ-effects). This is suggestive of the more general validity of Richardson's model at higher R_λ (unless intermittency effects make a significant difference at higher R_λ).

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