Simulating (2 + 1)-Dimensional Lattice QED with Dynamical Matter Using Ultracold Atoms

Erez Zohar,[1] J. Ignacio Cirac,[2] and Benni Reznik[1]

[1]*School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel-Aviv 69978, Israel*
[2]*Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, 85748 Garching, Germany*
(Received 6 September 2012; revised manuscript received 26 November 2012; published 29 January 2013)

We suggest a method to simulate compact quantum electrodynamics using ultracold atoms in optical lattices, which includes dynamical Dirac fermions in 2 + 1 dimensions. This allows us to test the dynamical effects of confinement as well as the deformations and breaking of two-dimensional flux loops, and to observe the Wilson-loop area law.

DOI: 10.1103/PhysRevLett.110.055302 PACS numbers: 67.85.Hj, 11.15.Ha

Recent progress in the manipulation and control of atomic systems has boosted the interest in the emerging field of quantum simulation [1–3]. So far, most of the theoretical and experimental effort has concentrated on simulating condensed matter systems. The field of quantum simulation, however, may also have a strong impact in high energy physics (HEP) [4], lending us the possibility of observing intriguing phenomena emerging from the standard model, or even allowing us to answer some questions which cannot be addressed with standard techniques of lattice quantum chromodynamics (QCD). In contrast to its condensed matter physics counterpart, the field of quantum simulation of HEP models is almost unexplored, and presents new theoretical and experimental challenges.

In the standard model of HEP, forces are mediated through gauge bosons, which are the excitations of gauge fields. Thus, gauge fields, either Abelian or non-Abelian, play a central role in particle physics. As such, gauge theories have been a subject of continuous research over the last decades, theoretically, numerically, and experimentally. One important property of such theories is known as confinement [5–8], which is manifested in real-world QCD, in the structure of hadrons: free quarks cannot be found in nature, due to the phenomenon of confinement, which "holds them together," forming hadrons. Confinement is known to hold in non-Abelian gauge theories (not only in QCD), but it is also manifested in Abelian theories, such as $U(1)$–quantum electrodynamics (QED): in the lattice case, compact QED (cQED). There, in (3 + 1) dimensions, a phase transition takes place between a confining phase (for strong couplings) and a Coulomb phase (for weak ones), and in (2 + 1) dimensions confinement takes place for all values of the coupling constant [7–11].

Recently, some suggestions for simulations of quantum field theory (QFT) and HEP models have been proposed. These include the simulations of dynamic scalar [12] (vaccum entanglement) and fermionic fields [13] (the interacting Thirring and Gross-Neveu models). Simulation proposals for fermions in lattice QFT, either free or in external nondynamical gauge fields, include axions and Wilson fermions [14], Dirac fermions in curved spacetime [15], and general quantum simulators of QFTs and topological insulators [16]. On the other hand, some simulations were proposed for dynamic gauge fields but with no fermions using BECs [17] or single atoms [18] in optical lattices, where the first simulates the Abelian Kogut-Susskind Hamiltonian [10] and the latter, a truncated ("spin-gauge") version of it (other examples of truncated models are given in Refs. [19–21]). More simulations of pure-gauge $U(1)$ theories with ultracold atoms have been recently suggested [22,23].

The next step is the inclusion of dynamical matter (fermions) in the model, allowing for simulation of full cQED. This is of special interest, both for the possibility of probing confinement in a dynamic matter theory, as well as for the exploration of problems which are not amenable to a numerical description due to the well-known Grassmanian integration sign problem [24]. The first proposals for the simulation of a fermionic system with a dynamic gauge field [13,25] are, so far, restricted to (1 + 1) dimensions, where the emerging physical phenomena are limited due to the absence of magnetic fields and of multiple paths connecting two points. A quantum simulation of QED in the continuum was suggested as well [26]. Compactness, which is a lattice feature absent in continuous theories, is essential for confinement [11].

In this Letter, we propose a spin-gauge model containing dynamic fermions that allows for simulation of nontrivial gauge field dynamics for (2 + 1)-dimensional cQED, including two-dimensional spatial effects such as flux loops. We show how to construct a gauge invariant model of a $U(1)$ gauge field coupled to 'naive' fermions, discuss how to create various initial states, and suggest several possible experiments, such as the observation of broken flux lines and manifestation of Wilson's area law of confinement [5], realizing the gedanken experiment proposed in Ref. [27].

The simulating system.—Consider a two-dimensional spatial optical lattice [3], filled with fermionic and bosonic atoms (see Fig. 1). Each vertex $\mathbf{n} = (n_1, n_2)$ can be occupied by fermions, either of species C or D, described by the local Hilbert spaces spanned by eigenstates of the

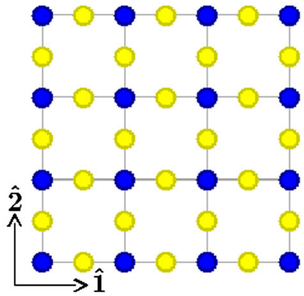

FIG. 1 (color online). The simulating lattice. Every link is occupied by a single boson [yellow (light) circles] and every vertex by either zero, one, or two fermions [blue (dark) circles].

fermionic number operators $N_{\mathbf{n}}^{C,D}$ (with the annihilation operators $c_{\mathbf{n}}$, $d_{\mathbf{n}}$). The two species form a spinor at each vertex:

$$\psi_{\mathbf{n}} = \begin{pmatrix} c_{\mathbf{n}} \\ d_{\mathbf{n}} \end{pmatrix}. \quad (1)$$

We define the local charge [28]

$$Q_{\mathbf{n}} = \psi_{\mathbf{n}}^{\dagger}\psi_{\mathbf{n}} - 1 = N_{\mathbf{n}}^{C} + N_{\mathbf{n}}^{D} - 1. \quad (2)$$

Each link (in directions $k = 1, 2$) is occupied by a single boson, of one of $2l + 1$ species, belonging to an angular momentum multiplet, described by eigenstates of $L_{z,\mathbf{n}}^{k}$ (see also Fig. 1).

We call this system "the primitive theory" because the gauge-invariant effective theory will arise from it by adding a constraint, as will be shown later, but first we shall describe the (separate) primitive theories for the bosons and the fermions. Similar fermionic theories can be found, for example, in Refs. [14–16].

The bosons primitive dynamics is governed by the Hamiltonian

$$H_p^b = \sum_{\mathbf{n},k}(\mu(L_{z,\mathbf{n}}^{k})^2 + 2\beta L_{x,\mathbf{n}}^{k}) + \Omega\sum_{\langle i,j\rangle}(L_{x,i}L_{x,j} + L_{y,i}L_{y,j}), \quad (3)$$

where the second sum is on nearest-neighbor (intersecting) links.

The fermions primitive dynamics is governed by the lattice 'naive' Dirac Hamiltonian

$$H_p^f = i\eta\sum_{\mathbf{n},k}(\psi_{\mathbf{n}}^{\dagger}\sigma_k\psi_{\mathbf{n}+\hat{\mathbf{k}}} - \text{H.c.}) + M\sum_{\mathbf{n}}\psi_{\mathbf{n}}^{\dagger}\sigma_z\psi_{\mathbf{n}}. \quad (4)$$

Imposing gauge invariance.—We wish to impose gauge invariance on the system. In order to do so, we constrain the generators of gauge symmetry $G_{\mathbf{n}}$ to zero by adding the "Gauss's Hamiltonian" H_G to the two primitive Hamiltonians. This Hamiltonian involves an interaction between the two primitive systems. The generators of local gauge transformations are

$$G_{\mathbf{n}} = L_{z,\mathbf{n}}^{1} + L_{z,\mathbf{n}}^{2} + L_{z,\mathbf{n}-\hat{\mathbf{1}}}^{1} + L_{z,\mathbf{n}-\hat{\mathbf{2}}}^{2} - (-1)^{n_1+n_2}Q_{\mathbf{n}}. \quad (5)$$

The constraint is implemented by the Gauss's Hamiltonian, $H_G = \lambda\sum_{\mathbf{n}}G_{\mathbf{n}}^2$, and thus λ must be the highest energy scale: $\lambda \gg \mu, \beta, \Omega, \eta, M$.

As λ is the largest energy scale, we wish to derive an effective theory for $H_G^{\prime}s$ ground sector [29], which will introduce gauge invariance to the system. In the first order, we obtain the bosonic "electric Hamiltonian" $H_E = \mu\sum_{\mathbf{n},k}(L_{z,\mathbf{n}}^{k})^2$, and the fermionic "mass Hamiltonian" $H_M = M\sum_{\mathbf{n}}\psi_{\mathbf{n}}^{\dagger}\sigma_z\psi_{\mathbf{n}}$.

In the second order, we obtain four contributions: (1) the bosonic "magnetic" plaquette term $H_B = -\frac{2\Omega^2}{\lambda}\sum_{\mathbf{n}}(L_{+,\mathbf{n}}^{1}L_{-,\mathbf{n}+\hat{\mathbf{1}}}^{2}L_{+,\mathbf{n}+\hat{\mathbf{2}}}^{1}L_{-,\mathbf{n}}^{2} + \text{H.c.})$ as well as the bosonic $H_B^{\prime}$, which has no equivalent in cQED, but is gauge invariant and contributes no interesting dynamics (see Ref. [18] for details), and (2) the "minimal coupling" Dirac terms $H_D = -\frac{i\beta\eta}{\lambda}\sum_{\mathbf{n},k}(\psi_{\mathbf{n}}^{\dagger}\sigma_k\psi_{\mathbf{n}+\hat{\mathbf{k}}}L_{s,\mathbf{n}}^{k} - \text{H.c.})$ (where $s = (-)^{n_1+n_2}$). The other two contributions are less important, and are described in Sec. 2 of the Supplemental Material [30].

Next we make the sign change $L_{z,\mathbf{n}}^{k} \to (-1)^{\mathbf{n}}L_{z,\mathbf{n}}^{k}$. Besides inverting the sign of L_z on odd links, we also swap the $L_{\pm}$ operators there (leaving L_x invariant). This results, as in the pure spin-gauge theory, with correct signs in the plaquette term

$$H_B = -\frac{2\Omega^2}{\lambda}\sum_{\mathbf{n}}(L_{+,\mathbf{n}}^{1}L_{+,\mathbf{n}+\hat{\mathbf{1}}}^{2}L_{-,\mathbf{n}+\hat{\mathbf{2}}}^{1}L_{-,\mathbf{n}}^{2} + \text{H.c.}), \quad (6)$$

and the gauge generators

$$G_{\mathbf{n}} = (-1)^{n_1+n_2}(L_{z,\mathbf{n}}^{1} + L_{z,\mathbf{n}}^{2} - L_{z,\mathbf{n}-\hat{\mathbf{1}}}^{1} - L_{z,\mathbf{n}-\hat{\mathbf{2}}}^{2} - Q_{\mathbf{n}}). \quad (7)$$

This also gives rise to the correct (minimally coupled) Dirac Hamiltonian

$$H_D = \frac{i\eta\beta}{\lambda}\sum_{\mathbf{n},k}(\psi_{\mathbf{n}+\hat{\mathbf{k}}}^{\dagger}\sigma_k\psi_{\mathbf{n}}L_{-,\mathbf{n}}^{k} - \text{H.c.}), \quad (8)$$

where we identify the Dirac matrices $\alpha_{1,2}$ as the Pauli matrices $\sigma_{1,2}$ (the Dirac β matrix is σ_z).

Finally, we introduce the gauge theory parameters g (coupling constant) and m (fermion mass) by rescaling the energy by $\alpha = \frac{2}{g^2}\mu = \frac{4l^2(l+1)^2\Omega^2g^2}{\lambda} = \frac{2\eta\beta\sqrt{l(l+1)}}{\lambda} = \frac{M}{m}$, and obtain

$$H_{l,\text{cQED}} = \alpha^{-1}(H_E + H_B + H_D + H_M). \quad (9)$$

This is the spin-gauge Hamiltonian of cQED with naive dynamic fermions.

We wish to emphasize that the previous effective Hamiltonian calculation does not depend on l; thus, if the primitive theory and the constraint are achieved for other values of l, the same effective Hamiltonian results.

Realization of the model.—In order to realize the required bosonic terms, one can use the methods presented in Ref. [18], which generalize those in Ref. [31]. For further details on that and on the fermionic interaction, see Sec. 1 of the Supplemental Material [30]. The main issue is now the realization of the boson-fermion interaction terms in H_G. The required terms, for each vertex $\mathbf{n}$, are of the form

$$\lambda\xi(\mathbf{n})(L^1_{z,\mathbf{n}} + L^2_{z,\mathbf{n}} + L^1_{z,\mathbf{n}-\hat{\mathbf{1}}} + L^2_{z,\mathbf{n}-\hat{\mathbf{2}}})(N^C_{\mathbf{n}} + N^D_{\mathbf{n}}), \tag{10}$$

where $\xi(\mathbf{n}) = -2(-1)^{n_1+n_2}$. For a start, we choose the atomic hyperfine levels of the fermionic degrees of freedom: the C and D fermions belong to the hyperfine levels $|F_C = \frac{1}{2}, m_C\rangle$, $|F_D = \frac{3}{2}, m_D\rangle$ respectively, where m_C, m_D are constant on each vertex. The hyperfine levels are arranged such that the other values of m, on each hyperfine manifold, are of a much larger energy, and thus are not accessible if they are not initially populated.

The terms in Eq. (10) are derived as scattering interactions between two particles: bosons with $F_b = 1$ and each of the fermionic species. The first quantized interaction Hamiltonian corresponding to such scattering processes is $H_{\rm sc} = \sum_F g_F P_F$, where the summation is on the total angular momentum of the two scattered particles, $\{g_F\}$ depend on the S-wave scattering length, and thus are tunable using optical Feshbach resonances [32–34], and P_F are projection operators to the subspaces of different total angular momenta. These operators can be built using the different values of $\mathbf{F}_C \cdot \mathbf{F}_b$, $\mathbf{F}_D \cdot \mathbf{F}_b$ for each value of total $\mathbf{F}$, and thus one can express the scattering Hamiltonians for C, D as

$$\begin{aligned} H^C_{\rm sc} &= \tilde{C}_0 + \tilde{C}_1 \mathbf{F}_C \cdot \mathbf{F}_b, \\ H^D_{\rm sc} &= \tilde{D}_0 + \tilde{D}_1 \mathbf{F}_D \cdot \mathbf{F}_b + \tilde{D}_2 (\mathbf{F}_D \cdot \mathbf{F}_b)^2, \end{aligned} \tag{11}$$

where $\tilde{C}_i = \tilde{C}_i(g_{\frac{1}{2}}, g_{\frac{3}{2}})$, $\tilde{D}_i = \tilde{D}_i(g_{\frac{1}{2}}, g_{\frac{3}{2}}, g_{\frac{5}{2}})$.

These Hamiltonians are next plugged into the second quantized interaction terms $\int d^3x \Psi^{C\dagger}_i \Phi^\dagger_j H^C_{\rm sc} \Psi^C_k \Phi_l$, where Ψ^C_k, Φ_l are the fermionic and bosonic second quantized wave functions (and similarly for D). Because of the energy spectrum of the fermions, the only possible processes are such with $i = k = m_C$. Thus no angular momentum transfer can take place for the bosons as well; hence, $j = l = m_b$, and only the z components of the angular momentum vectors contribute. We finally obtain, on each vertex,

$$\sum_{m_b} (C_0 + C_1 m_C m_b) c^\dagger_{\mathbf{n}} c_{\mathbf{n}} a^\dagger_{m_b} a_{m_b} + \sum_{m_b} [D_0 + D_1 m_D m_b + D_2 f(m_D, m_b)] d^\dagger_{\mathbf{n}} d_{\mathbf{n}} a^\dagger_{m_b} a_{m_b}, \tag{12}$$

where the bosonic operators correspond to each of the neighboring links, C_i, D_i are products of the coefficients $\tilde{C}_i$, $\tilde{D}_i$ and the appropriate overlap integrals, and $f(m_D, m_b)$ is quadratic.

In order to obtain Eq. (10), we impose conditions on the scattering coefficients g_F (controlled by Feshbach resonances) $D_2(g_{\frac{1}{2}}, g_{\frac{3}{2}}, g_{\frac{5}{2}}) = 0$, $D_0(g_{\frac{1}{2}}, g_{\frac{3}{2}}, g_{\frac{5}{2}}) = C_0(g_{\frac{1}{2}}, g_{\frac{3}{2}})$, and $D_1(g_{\frac{1}{2}}, g_{\frac{3}{2}}, g_{\frac{5}{2}})|m_D| = C_1(g_{\frac{1}{2}}, g_{\frac{3}{2}})|m_C| = -2\lambda$. The first condition eliminates the quadratic terms. The second condition results in terms of the form $C_0(N^C_{\mathbf{n}} + N^D_{\mathbf{n}})N^b$. However, the bosonic effective Hamiltonian sets $N^b = 1$ everywhere around the bosonic lattice, and thus this term is merely the total number of fermions in the system, which is an ignorable constant in the Hamiltonian. Setting at the vertex $\mathbf{n}$, $m_C = (-1)^{n_1+n_2}|m_C|$, $m_D = (-1)^{n_1+n_2}|m_D|$, we are only left with one type of term, which is, because of the third condition, $-2\lambda(-1)^{n_1+n_2}(N^C_{\mathbf{n}} + N^D_{\mathbf{n}})L_z$. This is the desired interaction.

No interactions vacuum and excited states.—Suppose first that the bosonic and fermionic interactions (H_B, H_D) are turned off, i.e., the system is pure gauge (the charges are static and uncoupled) and in the strong coupling limit. Hence the gauge field vacuum does not contain any excited links: all the bosonic atoms are in their $m = 0$ state. This is the exact cQED vacuum in the extreme strong limit, because no plaquette terms contribute.

Each vertex can contain four different fermionic contents. The state with no charges corresponds to the "Dirac sea," i.e., all the vertices are filled with D atoms. In such a state, the local "mass" is $-M$. Because energies are measured above the Dirac sea, we would like to measure these "masses" above $-M$, i.e., shift the energies with the product of M with the total number of vertices. This is the vacuum in the case of no charges. The three local fermionic excited states correspond to different simulated fermionic contents, as summarized in Table I (note that the masses are relative to $-M$).

Initial state preparation.—The system is initially prepared in the gauge+fermions ground state, which is an eigenstate of $H_G + H_E + H_M$,

$$|\text{vac}\rangle \equiv \bigotimes_{\text{links}} |m = 0\rangle \left(\bigotimes_{\text{vertices}} |0\rangle_C |1\rangle_D \right). \tag{13}$$

Then, these three Hamiltonians can be turned on, without changing the state but imposing the constraint.

There are several interesting initial states one can obtain, even before turning on the field-matter interaction H_D. These include (a) the QED static vacuum (up to first order,

TABLE I. Fermionic contents of the vertices.

Real content	"Mass"	"Charge"	Simulated content
$\lvert 0\rangle_C \lvert 1\rangle_D$	0	0	Empty vertex
$\lvert 0\rangle_C \lvert 0\rangle_D$	M	-1	$\bar{q}$
$\lvert 1\rangle_C \lvert 1\rangle_D$	M	$+1$	q
$\lvert 1\rangle_C \lvert 0\rangle_D$	$2M$	0	$q\bar{q}$

if one works with $l = 1$) obtained by increasing Ω adiabatically, keeping $\frac{2\Omega^2 l^2(l+1)^2}{\lambda} \ll \mu$, (b) a "loop sea" obtained by increasing Ω adiabatically again, but with $\frac{2\Omega^2 l^2(l+1)^2}{\lambda} \gg \mu$, dressing to a state with many loops, but with larger amplitudes, and then, lowering the value of Ω to obtain strong regime QED dynamics again, and (c) zeroth-order excitations created using single-addressing lasers [35,36]: large closed flux loops as in the pure gauge case [see Fig. 2(a)] and/or mesons, i.e., C charges and empty vertices with zeroth-order flux tubes connecting them. Afterwards H_B can be turned on adiabatically to obtain QED dynamics.

Inclusion of matter dynamics.—Given any of the initial states (a), (b), or (c), the field-matter interaction H_D can be turned on (in order to remain in the strong limit, the parameters must obey $\frac{2\Omega^2 l^2(l+1)^2}{\lambda} \ll \frac{\eta\beta\sqrt{l(l+1)}}{\lambda} \ll \mu$ (see Sec. 3 of the Supplemental Material [30]). Then, an interplay between the fermion masses and the fermionic interaction terms introduces changes in the flux loops or tubes structure, if one waits long enough. For example, suppose $2M = \mu L$, for some integer L. Then, if there is a flux tube or loop longer than L, we expect it to break with some probability to two flux tubes, generating two new fermions, and transferring the energy of the "disappearing links" to the masses of the new fermions. On the other hand, if the fermion mass is small (or zero) we expect the flux tube to break and form new fermions constantly.

Another possibility is to slowly move (externally) one of the charges of an initially prepared flux-tube and stretch it, which should lead to it breaking apart.

Examples of matter dynamic processes are shown in Fig. 2.

Wilson's area law observation.—Another possible measurement which can be done in this system is a test of the Wilson-loop area law (dependence on an area of a space-time rectangle) [5] as a probe for confinement. This can be done by an interference of two mesons in superposition. In real-world QED, however, it is only a gedanken experiment [27]; here we propose to realize it in the quantum simulation. For that, we need the fermions to be static and their mass is unnecessary, and hence H_M, H_f are turned off (and thus, effectively, H_D as well). In fact, for a proof of principle of this experiment, H_B is also not needed.

We start, in the extreme strong limit, with a mesonic state: a single flux-tube with length R over the Dirac sea $|R\rangle$, emanating between the vertices (m, n) and $(m + R, n)$, such that $Q_{m,n} = 1$, $Q_{m+R,n} = -1$. As a meson in the strong coupling limit, it is an eigenstate of the Hamiltonian. Then, as in Ref. [27], we wish to create a superposition of two states, $|R\rangle$ and $|R + 1\rangle$: i.e., transfer the C fermion, with a probability of $\frac{1}{2}$, from (m, n) to $(m - 1, n)$. We shall treat this subspace as a two-level system, denoted by $|\downarrow\rangle = |R\rangle$, $|\uparrow\rangle = |R + 1\rangle$, with the Hamiltonian $H_0 = \mu|\uparrow\rangle\langle\uparrow|$.

The required superposition is generated as in Ramsey interferometry (see Sec. 5 of the Supplemental Material [30]): we apply on the initial state $|\downarrow\rangle$ the unitary operation $U = e^{-i(\pi/4)\sigma_y}$, and obtain the state $U|\downarrow\rangle = -|\downarrow_x\rangle$. Next, we wait a long time $T \gg \tau$, during which the state evolves to $\frac{1}{\sqrt{2}}e^{-i\mu RT}(|\downarrow\rangle - e^{-i\mu T}|\uparrow\rangle)$. Operating on the state with U again yields $\frac{1}{\sqrt{2}}(|\downarrow_x\rangle + e^{-i\mu T}|\uparrow_x\rangle)$ (neglecting the global phase). Switching back to the σ_z basis, recalling the definitions of our effective two-level system, we obtain that the probabilities of finding a fermion C in the vertices (m, n), $(m - 1, n)$ are

$$P_{m,n} = \sin^2\left(\frac{\mu LT}{2}\right), \qquad P_{m-1,n} = \cos^2\left(\frac{\mu LT}{2}\right). \qquad (14)$$

Because $A = LT$ (here $L = 1$), this manifests the area law in a confining phase (see Fig. 3); thus, within this system a realization of the proposed gedanken experiment is possible. In future generalizations, this method may serve as a phase transition probe.

Summary.—In this Letter, we have proposed a method to add dynamic fermions to the spin-gauge model introduced in Ref. [18]. This allows us to simulate

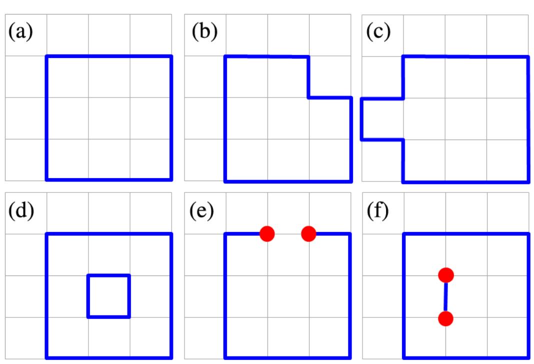

FIG. 2 (color online). Examples of gauge field and matter dynamics, as explained in the text, for an initial state of a single flux loop (a). Panels (b)–(f) show possible outcomes of first-order processes. Panels (b)–(d) result from operations of H_B: (b) loop decreasing, (c) loop increasing, (d) exciting a non-connected plaquette. Panels (e)–(f) result from operations of H_D: (e) loop breaking and creation of two charges, (f) creation of a meson outside the loop.

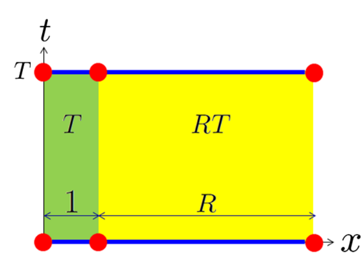

FIG. 3 (color online). Description of the area-law experiment in space-time, as described in the text. The interference phase depends on the green (darker) area.

(2 + 1)-dimensional cQED using ultracold atoms in optical lattices, enabling the observation of confinement in a dynamic matter theory, and suggests a detour to the well known Grassmanian integration sign problem, encountered in Monte Carlo simulations [24]. Simulations of our model on a small lattice may be used to check the effects of imperfections of the parameters on the dynamics. We have suggested several possible measurements, which can show the emergence of dynamic charges, the breaking of flux-tubes, and the area law behavior of the confining potential.

B. R. acknowledges the support of the Israel Science Foundation, the German-Israeli Foundation, and the European Commission (PICC). I. C. is partially supported by the EU project AQUTE. E. Z. acknowledges the support of the Adams Fellowship of the Israel Academy of Sciences and Humanities.

[1] R. Blatt and C. F. Roos, Nat. Phys. **8**, 277 (2012).
[2] I. Bloch, J. Dalibard, and S. Nascimbene, Nat. Phys. **8**, 267 (2012).
[3] M. Lewenstein, A. Sanpera, and V. Ahufinger, *Ultracold Atoms in Optical Lattices: Simulating Quantum Many-Body Systems* (Oxford University Press, New York, 2012).
[4] J. I. Cirac and P. Zoller, Nat. Phys. **8**, 264 (2012).
[5] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[6] J. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975).
[7] A. M. Polyakov, Nucl. Phys. **B120**, 429 (1977).
[8] T. Banks, R. Myerson, and J. Kogut, Nucl. Phys. **B129**, 493 (1977).
[9] S. D. Drell, H. R. Quinn, B. Svetitsky, and M. Weinstein, Phys. Rev. D **19**, 619 (1979).
[10] J. B. Kogut, Rev. Mod. Phys. **51**, 659 (1979).
[11] S. Ben-Menahem, Phys. Rev. D **20**, 1923 (1979).
[12] A. Retzker, J. I. Cirac, and B. Reznik, Phys. Rev. Lett. **94**, 050504 (2005).
[13] J. I. Cirac, P. Maraner, and J. K. Pachos, Phys. Rev. Lett. **105**, 190403 (2010).
[14] A. Bermudez, L. Mazza, M. Rizzi, N. Goldman, M. Lewenstein, and M. A. Martin-Delgado, Phys. Rev. Lett. **105**, 190404 (2010).
[15] O. Boada, A. Celi, J. I. Latorre, and M. Lewenstein, New J. Phys. **13**, 035002 (2011).
[16] L. Mazza, A. Bermudez, N. Goldman, M. Rizzi, M. A. Martin-Delgado, and M. Lewenstein, New J. Phys. **14**, 015007 (2012).
[17] E. Zohar and B. Reznik, Phys. Rev. Lett. **107**, 275301 (2011).
[18] E. Zohar, J. I. Cirac, and B. Reznik, Phys. Rev. Lett. **109**, 125302 (2012).
[19] D. Horn, Phys. Lett. **100B**, 149 (1981).
[20] P. Orland and D. Rohrlich, Nucl. Phys. **B338**, 647 (1990).
[21] S. Chandrasekharan and U.-J. Wiese, Nucl. Phys. **B492**, 455 (1997).
[22] G. Szirmai, E. Szirmai, A. Zamora, and M. Lewenstein, Phys. Rev. A **84**, 011611 (2011).
[23] L. Tagliacozzo, A. Celi, A. Zamora, and M. Lewenstein, arXiv:1205.0496v1.
[24] M. Troyer and U.-J. Wiese, Phys. Rev. Lett. **94**, 170201 (2005).
[25] D. Banerjee, M. Dalmonte, M. Müller, E. Rico, P. Stebler, U.-J. Wiese, and P. Zoller, Phys. Rev. Lett. **109**, 175302 (2012).
[26] E. Kapit and E. Mueller, Phys. Rev. A **83**, 033625 (2011).
[27] E. Zohar and B. Reznik, arXiv:1208.1012.
[28] Note that this definition of charges is similar to the charge in the second quantized Dirac field. There, for the lower entry of the spinor, we swap the fermionic creation and annihilation operators, and thus obtain that this charge transforms to the $N_C - N_D$.
[29] C. E. Soliverez, J. Phys. C **2**, 2161 (1969).
[30] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.110.055302 for more technical details.
[31] L. Mazza, M. Rizzi, M. Lewenstein, and J. I. Cirac, Phys. Rev. A **82**, 043629 (2010).
[32] P. O. Fedichev, Y. Kagan, G. V. Shlyapnikov, and J. T. M. Walraven, Phys. Rev. Lett. **77**, 2913 (1996).
[33] J. L. Bohn and P. S. Julienne, Phys. Rev. A **56**, 1486 (1997).
[34] F. K. Fatemi, K. M. Jones, and P. D. Lett, Phys. Rev. Lett. **85**, 4462 (2000).
[35] W. S. Bakr, J. I. Gillen, A. Peng, S. Folling, and M. Greiner, Nature (London) **462**, 74 (2009).
[36] C. Weitenberg, M. Endres, J. F. Sherson, M. Cheneau, P. Schausz, T. Fukuhara, I. Bloch, and S. Kuhr, Nature (London) **471**, 319 (2011).