Integer Quantum Hall Effect for Bosons

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A simple physical realization of an integer quantum Hall state of interacting two dimensional bosons is provided. This is an example of a symmetry-protected topological (SPT) phase which is a generalization of the concept of topological insulators to systems of interacting bosons or fermions. Universal physical properties of the boson integer quantum Hall state are described and shown to correspond with those expected from general classifications of SPT phases.

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Introduction.—Consider a system of two dimensional bosons with particle number conservation in the absence of time reversal symmetry. Can such a system form a gapped phase that is qualitatively different from a conventional Mott insulator, but has no "intrinsic" topological order—i.e., no fractional excitations and a unique ground state on topologically nontrivial manifolds? Recent work shows that the answer is yes. In fact, according to the powerful cohomology classification scheme of Ref. [1], there are infinite number of such phases, with each phase labeled by an integer $n \neq 0$.

More generally, Ref. [1] proposed a classification scheme for bosonic phases with *arbitrary symmetry* (time reversal, particle number conservation, etc.) and no intrinsic topological order. These phases have been called symmetry-protected topological (SPT) phases and can be regarded as generalizations of integer quantum Hall states and topological insulators to interacting systems of either bosons or fermions. Loosely speaking, SPT phases are characterized by the fact that their ground state wave functions have short-range entanglement, but are nevertheless distinct from a product state, such as a conventional Mott insulator. More physically, SPT phases are distinguished by the presence of robust edge modes that cannot be gapped out or localized unless the relevant symmetry is broken [2,3].

Very recently Lu and Vishwanath [4] provided a beautifully simple discussion of such symmetry-protected topological phases in two dimensions in terms of a Chern-Simons approach and a classification of the associated K matrices in the presence of symmetries. (A similar analysis was given in Ref. [5] for the case of time reversal and charge conservation symmetry.) Their description gives easy access to the universal properties of such phases. For the specific case we consider here, namely bosons with a $U(1)$ charge conservation symmetry [6], Ref. [4] showed that the integer that labels these phases can be physically interpreted in terms of a quantized electric Hall conductivity. Specifically, the phase labeled by n has an electric Hall conductivity of $\sigma_{xy} = 2n$ in appropriate units. Hence, these phases can be thought of as integer quantum Hall states for bosons.

Here we propose a simple physical system where the simplest integer quantum Hall state may be realized. (An alternative realization using a coupled wire construction is discussed in Lu and Vishwanath [4]). Specifically we argue that a system of two-component bosons in a strong magnetic field may admit a stable integer quantum Hall phase. A natural realization is in terms of pseudospin-$1/2$ "spinor" bosons of ultracold atoms in artificial gauge fields. We analyze the basic physical properties of this state and show that they agree with the results expected from the general classification of Refs. [1,4]. In particular, the particle number Hall conductivity is quantized to be 2 while the thermal Hall conductivity is quantized to 0. This is related to the presence of two branches of counterpropagating chiral edge modes—one that carries particle current and one that is neutral—that are protected by the global charge $U(1)$ symmetry. As a bonus we show that when pseudospin $SU(2)$ symmetry is present, the gapless edges cannot be gapped even if boson number conservation is explicitly broken. Thus, in this situation, this state may equally well be viewed as an example of an $SU(2)$ symmetric SPT state, which also is predicted to occur by the classification of Ref. [1].

An important limitation of our analysis is that we do not identify a *specific* microscopic Hamiltonian that realizes the integer quantum Hall phase. Instead, we describe a general class of systems where these states may occur. A particularly simple example is one where the bosons interact just through delta-function repulsion. We, however, leave questions of energetics to future work.

The model.—We consider a two-component system of bosons (for instance, spinor bosons or a bilayer system) in a strong magnetic field B such that each component is at filling factor $\nu = 1$. Initially we assume that there is no interspecies tunneling but will relax this assumption later. Without interspecies tunneling, the system actually has $U(1) \times U(1)$ symmetry corresponding to separate conservation of the two species of bosons. The Hamiltonian is

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$$H = \sum_I H_I + H_{\rm int}, \tag{1}$$

$$H_I = \int d^2x b_I^\dagger \left(-\frac{(\vec{\nabla} - i\vec{A})^2}{2m} - \mu\right) b_I, \tag{2}$$

$$H_{\rm int} = \int d^2x d^2x' \rho_I(x) V_{IJ}(x - x') \rho_J(x'). \tag{3}$$

Here b_I is the boson annihilation operator for species I where $I = 1, 2$ and $\rho_I = b_I^\dagger b_I$ is the corresponding boson density. The vector potential $\vec{A}$ describes the external B field. We assume that the interactions V_{IJ} are short ranged and repulsive.

Depending on the detailed form of the interactions, a number of different states may be realized by this system. Here we focus on a particular candidate state that corresponds to the integer quantum Hall phase discussed above. Later we will discuss some of the other possible competing phases.

To construct our candidate state, we use a flux attachment Chern-Simons theory. We define new boson operators

$$\tilde{b}_1(x) = e^{-i\int d^2x' \Theta(x-x')\rho_2(x')} b_1(x) \tag{4}$$

$$\tilde{b}_2(x) = e^{-i\int d^2x' \Theta(x-x')\rho_1(x')} b_2(x), \tag{5}$$

where $\Theta(x)$ is the angle at which the vector $\vec{x}$ points. This implements a flux attachment where each boson is attached to one flux quantum of the other species. We will call the bosons $\tilde{b}_{1,2}$ "mutual composite bosons." With $\nu = 1$ for each species, we can clearly cancel the flux of the external magnetic field in a flux smearing mean field approximation. Following the usual quantum Hall logic, an effective Chern-Simons Landau-Ginzburg theory may be written down in terms of these mutual composite bosons and takes the form

$$\begin{aligned}
\mathcal{L} &= \sum_I \mathcal{L}_I + \mathcal{L}_{\rm int} + \mathcal{L}_{\rm CS} \\
\mathcal{L}_I &= i\tilde{b}_I^*(\partial_0 - iA_{I0} + i\alpha_{I0})\tilde{b}_I \\
&\quad - \frac{|\vec{\nabla}\tilde{b}_I - i(\vec{A}_I - \vec{\alpha}_I)\tilde{b}_I|^2}{2m} + \mu|\tilde{b}_I|^2 \\
\mathcal{L}_{\rm int} &= -V_{IJ}|\tilde{b}_I|^2|\tilde{b}_J|^2 \\
\mathcal{L}_{\rm CS} &= \frac{1}{4\pi}\epsilon^{\mu\nu\lambda}(\alpha_{1\mu}\partial_\nu\alpha_{2\lambda} + \alpha_{2\mu}\partial_\nu\alpha_{1\lambda}).
\end{aligned} \tag{6}$$

Here we have introduced two gauge fields α_1 and α_2 coupled by a mutual Chern-Simons term to implement the flux attachment. For convenience we have also introduced external probe gauge fields A_I, which couple to the boson currents of species I.

As the mutual composite bosons see zero average flux, we can imagine a situation in which they condense. This will lock the internal gauge field α_I to the probe external field A_I. The effective Lagrangian for the probe gauge fields then becomes

$$\mathcal{L}_{\rm eff} = \frac{1}{4\pi}\epsilon^{\mu\nu\lambda}(A_{1\mu}\partial_\nu A_{2\lambda} + A_{2\mu}\partial_\nu A_{1\lambda}). \tag{7}$$

Let us now define new probe gauge fields that couple to the total charge and pseudospin currents: $A_c = (A_1 + A_2)/2$, $A_s = (A_1 - A_2)/2$. In terms of these fields,

$$\mathcal{L}_{\rm eff} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}(A_{c\mu}\partial_\nu A_{c\lambda} - A_{s\mu}\partial_\nu A_{s\lambda}). \tag{8}$$

It follows that this state is incompressible and has a quantized electric Hall conductivity of $\sigma_{xy} = +2$ in appropriate units. It is thus an integer quantum Hall state of bosons. We can also see that this state has a pseudospin Hall conductivity of -2. However, this quantity is less robust as pseudospin conservation can be broken by inclusion of interspecies tunneling.

Several implications follow from the nonzero value for the electric Hall conductivity. First, we can see that the above quantum Hall state belongs to a different phase from the conventional Mott insulator (whose Hall conductivity vanishes). Second, we conclude that the above system has robust gapless edge modes, which cannot be gapped out unless charge conservation symmetry is broken (either explicitly or spontaneously).

For certain purposes, it is useful to describe this state using the K-matrix formalism for Abelian Chern-Simons theory. Starting from the Chern-Simons term Eq. (6), it is not hard to show that the above state corresponds to a K matrix

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and a charge vector $t^T = (1, 1)$. An important consequence of this identification is that the system does not support quasiparticle excitations with fractional charge or fractional statistics; in general such fractionalized excitations require a K matrix with $|\det(K)| > 1$.

To obtain a model for the edge modes, we use the bulk-edge correspondence for Abelian Chern-Simons theory. According to this correspondence, the edge theory corresponding to the above K matrix is given by

$$\mathcal{L} = \frac{1}{4\pi}(\partial_x\phi_1\partial_t\phi_2 + \partial_x\phi_2\partial_t\phi_1 - v_{IJ}\partial_x\phi_I\partial_x\phi_J), \tag{9}$$

where $\frac{1}{2\pi}\partial_x\phi_I$ describes the density of bosons in layer I, and where v_{IJ} is the velocity matrix. Diagonalizing the above action, it is easy to check that the edge contains two counterpropagating chiral modes—one of which carries electric charge, and one of which is electrically neutral (but carries pseudospin). As a result of this structure, the thermal Hall conductivity vanishes, even though the electric Hall conductivity is nonzero. We note that the vanishing of the thermal Hall conductivity distinguishes the above integer quantum Hall state from another class of unfractionalized bosonic phases that require no symmetry at all—namely the phases discussed in Refs. [4,7]. These phases have a nonvanishing thermal Hall conductivity which is a multiple of 8 (in appropriate units).

At a microscopic level, we can understand the stability of the edge as arising from the fact that backscattering between the two counterpropagating modes is prohibited by the $U(1)$ charge conservation symmetry. In this sense, the edge modes are symmetry-protected. More generally, edge reconstruction may modify the above picture, but properties such as the Hall response and overall edge stability are universal.

Wave function.—It is tempting but incorrect to guess that the ground state wave function for the above bosonic quantum Hall state is simply

$$\Psi(\{z_i, w_j\}) = \prod_{i,j}(z_i - w_j)e^{-\sum_i(|z_i|^2+|w_i|^2)/4}, \qquad (10)$$

where z_i, w_i label the complex spatial coordinates of the two species of bosons. In the standard Halperin notation for bilayer quantum Hall states, this is a (001) state of bosons. This wave function is unstable to spontaneous phase separation, as is readily seen using the plasma analogy: one can check that the plasma has attractive logarithmic interactions between the pseudospin densities, which implies a spontaneous ordering of the pseudospin density, i.e., phase separation. A modified wave function, which describes a uniform boson integer quantum Hall state, may be written down as

$$\Psi_{\mathrm{mod}} = \prod_{i<j}|z_i - z_j|\prod_{i<j}|w_i - w_j| \times \prod_{i,j}\frac{(z_i - w_j)}{|z_i - w_j|}e^{-\sum_i(|z_i|^2+|w_i|^2)/4}. \qquad (11)$$

In the plasma analogy this wave function has the same effective Hamiltonian as the (110) state and hence describes a uniform density fluid with $\nu = 1$ for either species. Indeed, a direct derivation of the wave function from the composite boson theory along the lines of Ref. [8] yields precisely Ψ_{mod}.

Alternatively, we can construct a ground state wave function using a mean field flux attachment procedure similar to the one used in Ref. [9] to construct a spin-singlet $\nu = 2/3$ state (see also Ref. [10]). First, we imagine attaching -1 flux quanta to each boson, thereby transforming the system into a bilayer of composite fermions at filling $1/2 \oplus 1/2$. We then imagine that the composite fermions form a (111) state. Finally, we project onto the lowest Landau level. The resulting wave function is

$$\Psi_{\mathrm{flux}} = P_{\mathrm{LLL}}\prod_{i<j}|z_i - z_j|^2\prod_{i<j}|w_i - w_j|^2 \times \prod_{i,j}(z_i - w_j)e^{-\sum_i(|z_i|^2+|w_i|^2)/4}, \qquad (12)$$

where P_{LLL} denotes the projection onto the lowest Landau level.

An interesting feature of the two wave functions, Eqs. (11) and (12), is that they are spin singlets under the $SU(2)$ pseudospin symmetry. (One way to see this is to note that, before projection, both wave functions can be written as a product of the antianalytic (221) state and a fully symmetric function of z_i, w_j. Given that the (221) state is a spin singlet, it follows that both of the above wave functions are also spin singlets.) This enhanced symmetry means that we can equally well regard the bosonic integer quantum Hall state as an example of an SPT phase with $SU(2)$ pseudospin symmetry, rather than $U(1)$ particle number conservation symmetry. In particular, explicit breaking of the global $U(1)$ symmetry by adding spin singlet pairing terms to the Hamiltonian will not gap out the edges so long as pseudospin $SU(2)$ symmetry is preserved. Thus either of these two symmetries will lead to protected gapless edge modes, due to the nonzero electric and pseudospin Hall conductivities, respectively.

Competing states.—We now discuss some of the possible competitors to the above integer quantum Hall state. One possibility is phase separation: the two species of bosons may clump together in different spatial regions. Such phase separated states are particularly natural in the limit where the interspecies interaction V_{12} is large compared with the same-species interactions V_{11}, V_{22}. Interestingly, a fully phase separated state has $\nu = 2$ in each puddle and therefore may realize a $k = 4$ non-Abelian Read-Rezayi state [11,12]. Another potential competitor is a decoupled state where the two species of bosons form uncorrelated $\nu = 1$ states. Such decoupled states are likely to be realized in the limit where V_{12} is small compared with V_{11}, V_{22}. This possibility is also potentially interesting because bosons at $\nu = 1$ may form a $k = 2$ non-Abelian Read-Rezayi state that is just the familiar Moore-Read Pfaffian state [11]. A third competitor is the $\nu = 2$ non-Abelian spin singlet state of Ardonne and Schoutens [13]. This state is a good candidate at or near the $SU(2)$ symmetric point where $V_{12} = V_{11} = V_{22}$.

Determining the specific circumstances under which the bosonic integer quantum Hall state wins out over its competitors is a detailed energetics question that will not be attempted here. We simply note that the integer quantum Hall state is a reasonable candidate in the regime where the interspecies interaction V_{12} is comparable to the same-species interactions V_{11}, V_{22}. We mention in passing that very recent numerical work [14,15] suggests that, in the case of delta-function repulsive interactions, the ground state at the $SU(2)$ symmetric point is a gapped spin singlet. Among other possibilities, this state could be the integer quantum Hall state discussed here, or the non-Abelian spin singlet state of Refs. [13,16].

An obvious experimental context to seek a realization of this phase is in ultracold atoms in artificial gauge fields.

In that context, instead of a bilayer it will be simpler to use spinor bosons and let the boson spin play the role of the bilayer index. Given the large number of interesting competing phases in this system, it seems worthwhile to explore experimentally the phases of spinor bosons at a total filling factor $\nu = 2$.

Nonlinear sigma model description.—It is interesting to view the boson integer quantum Hall state from a different point of view. As is well known [17] the Bose condensate phase of two-component bosons is described by an $SU(2)$ matrix order parameter (with, in general, $O(2) \times O(2)$ anisotropy). To be specific, write the fields $b_{1,2}$ in terms of their real and imaginary parts $b_1 = b_{1r} + ib_{1i}$, $b_2 = b_{2r} + ib_{2i}$, and restrict them to the surface $|b_1|^2 + |b_2|^2 = 1$. This can be organized as an $SU(2)$ matrix $g = b_{1r} + ib_{1i}\tau^z + ib_{2r}\tau^x + ib_{2i}\tau^y$ where $\vec{\tau}$ are the usual Pauli matrices. It is clear that the charge $U(1)$ symmetry acts by right multiplication by $e^{i\tau^z\chi}$ while the pseudospin $U(1)$ is generated by left multiplication by $e^{i\tau^z\chi}$. [Full pseudospin $SU(2)$ rotation symmetry, if present, is realized as left multiplication by an $SU(2)$ matrix.]

It is natural to attempt a description of the phases of the two-component boson system in terms of a quantum nonlinear sigma model based on this $SU(2)$ matrix order parameter. The Bose condensed phase of course has $\langle g \rangle \neq 0$. Disordered phases where $\langle g \rangle = 0$ correspond to the strong coupling limit of such a nonlinear sigma model. In general, the effective nonlinear sigma model for an $SU(2)$ matrix-valued order parameter in two space dimensions admits an interesting topological θ term corresponding to $\pi_3(SU(2)) = Z$.

We now argue that when $\theta = 2\pi$, this sigma model describes an bosonic integer quantum Hall phase, while when $\theta = 0$, it describes a trivial boson insulator (e.g., an ordinary boson Mott insulator). One way to see this is to analyze the edges of the two systems. First consider a sigma model with $\theta = 2\pi$ and full $SO(4) \sim SU(2)_R \times SU(2)_L$ symmetry. In this case, it is known that the boundary to the vacuum is described by a $1 + 1$ dimensional $SU(2)$ level-1 Wess-Zumino-Witten model [18,19] where the $SU(2)_R$ and $SU(2)_L$ currents move in opposite directions. Introducing an $O(2) \times O(2)$ anisotropy, we obtain an edge structure with an electrically charged edge mode moving in one direction and a pseudospin edge mode moving in the opposite direction. This structure agrees with the bosonic integer quantum Hall edge theory Eq. (9). Likewise, when $\theta = 0$, the boundary of the sigma model is presumably gapped—in agreement with the trivial boson insulator edge. We interpret these matching edge theories as evidence for the identifications claimed above.

The topological nonlinear sigma model at $\theta = 2\pi$ also plays a crucial role in the cohomology classification [1]. Thus the discussion in this section provides a connection between the K-matrix description of the boson integer quantum Hall state and the cohomology classification. Recently, simulations of $O(2) \times O(2)$ models with θ terms have appeared [20,21]. It should be interesting to examine disordered phases of models similar to these near $\theta = 2\pi$ and study their boundary to $\theta = 0$ insulators.

Discussion.—In this Letter, we have constructed an integer quantum Hall state for bosons with an electric Hall conductivity of $\sigma_{xy} = 2$ (in appropriate units). It is natural to wonder whether it is possible to construct a more "elementary" integer quantum Hall state—that is, a state with $\sigma_{xy} = 1$. We now argue that such a state is impossible if the system does not support fractional quasiparticle excitations. To see this, consider a general bosonic quantum Hall state, and imagine puncturing it at some point z_0 and adiabatically inserting 2π flux through the hole. This operation will create an excitation at z_0 with charge σ_{xy}. Let us consider the braiding statistics of these excitations. If we braid one excitation around another, the statistical phase follows from the Aharonov-Bohm effect: $\theta = 2\pi\sigma_{xy}$. Similarly, if we exchange two particles, the associated phase is $\theta/2 = \pi\sigma_{xy}$. On the other hand, if the state does not support fractional quasiparticles, then these excitations (like all other quasiparticle excitations) must be bosons. We conclude that σ_{xy} must be even for any bosonic quantum Hall state without fractional quasiparticle excitations.

We expect that the construction in this Letter can be extended to other symmetry-protected topological phases, such as bosonic topological insulators, which are protected by time reversal and charge conservation symmetry, and bosonic phases, which are protected by a discrete Z_n symmetry. The boson integer quantum Hall state described here provides a very simple prototypical example of these phases that, in addition, may be realizable in future experiments.

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