Fermion-Parity Anomaly of the Critical Supercurrent in the Quantum Spin-Hall Effect

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The helical edge state of a quantum spin-Hall insulator can carry a supercurrent in equilibrium between two superconducting electrodes (separation L, coherence length ξ). We calculate the maximum (critical) current I_c that can flow without dissipation along a single edge, going beyond the short-junction restriction $L \ll \xi$ of earlier work, and find a dependence on the fermion parity of the ground state when L becomes larger than ξ. Fermion-parity conservation doubles the critical current in the low-temperature, long-junction limit, while for a short junction I_c is the same with or without parity constraints. This provides a phase-insensitive, dc signature of the 4π-periodic Josephson effect.

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The quantum Hall effect and quantum spin-Hall effect both refer to a two-dimensional semiconductor with an insulating bulk and a conducting edge, and both exhibit a quantized electrical conductance between two metal electrodes. If the electrodes are superconducting, a current can flow in equilibrium, induced by a magnetic flux without any applied voltage. In the quantum Hall effect, the edge states are chiral (propagating in a single direction only) and two opposite edges are needed to carry a supercurrent [1–3]. Graphene is an ideal system to study this interplay of the Josephson effect and the quantum Hall effect in a strong magnetic field [4–6].

The interplay of the Josephson effect and the quantum spin-Hall effect, in zero magnetic field, has not yet been demonstrated experimentally but promises to be strikingly different [7]. The quantum spin-Hall insulator has helical edge states (propagating in both directions) that can carry a supercurrent along a single edge. The edge state couples a pair of Majorana zero modes, allowing for the transmission of unpaired electrons with h/e rather than $h/2e$ periodic dependence on the magnetic flux [8,9].

An h/e flux periodicity corresponds to a 4π periodicity in terms of the superconducting phase difference ϕ, which means that the current-phase relationship has two branches $I_\pm(\phi)$ and the system switches from one branch to the other when ϕ is advanced by 2π at a fixed total number $\mathcal{N}$ of electrons in the system. This is referred to as a fermion-parity anomaly, because the two branches have different parity $\sigma = \pm$ of the number of electrons in the superconducting ground state [8].

Josephson junctions come in two types [10], depending on whether the separation L of the superconducting electrodes is small or large compared to the coherence length $\xi = \hbar v/\Delta$, or equivalently, whether the superconducting gap Δ is small or large compared to the Thouless energy $E_{\rm T} = \hbar v/L$. Existing literature [7–9,11–18] has focused on the short-junction regime $L \ll \xi$. The supercurrent is then determined entirely by the phase dependence of a small number of Andreev levels in the gap, just one per transverse mode. The phase dependence of the continuous spectrum above the gap can be neglected. As the ratio L/ξ increases, the Andreev levels proliferate and also the continuous spectrum starts to contribute to the supercurrent. Since σ is switched by changing the occupation of a single level, one might wonder whether a significant parity dependence remains in the long-junction regime.

Remarkably enough, the parity dependence becomes even stronger. While in a short junction the two branches $I_+(\phi) = -I_-(\phi)$ differ only in sign, we find that in a long junction they differ both in sign and in magnitude. In particular, the largest current that can flow without dissipation is twice as large for I_- as it is for I_+. The difference is illustrated in Fig. 1, in the zero-temperature limit. The basic physics can be explained in simple terms, as we will do first, and then we will present a complete theory for a finite temperature and for an arbitrary ratio L/ξ.

We set the stage by summarizing the findings of Fu and Kane [7] in the short-junction regime. The spectrum of the Bogoliubov–de Gennes Hamiltonian $H_{\rm BdG}$ is a $\pm\varepsilon$ symmetric combination of a discrete spectrum for $|\varepsilon| < \Delta$ and a continuous spectrum for $|\varepsilon| > \Delta$. Since backscattering along the quantum spin-Hall edge is forbidden by time-reversal symmetry [19], this is a ballistic single-channel Josephson junction. In the limit $L/\xi \to 0$ the discrete spectrum consists of a pair of levels at $\varepsilon_\pm = \mp\Delta|\cos(\phi/2)|$, while the continuous spectrum is ϕ independent [20]. Quite generally, an eigenvalue $\varepsilon(\phi)$ of $H_{\rm BdG}$ contributes to the supercurrent an amount

$$I(\phi) = \frac{ge}{\hbar}\frac{d}{d\phi}\varepsilon(\phi), \tag{1}$$

with g a factor that counts spin and other degeneracies [21]. There is no spin degeneracy at the quantum spin-Hall edge (since spin is tied to the direction of motion), so $g = 1$ and the level $\varepsilon_\pm$ contributes a supercurrent [7]

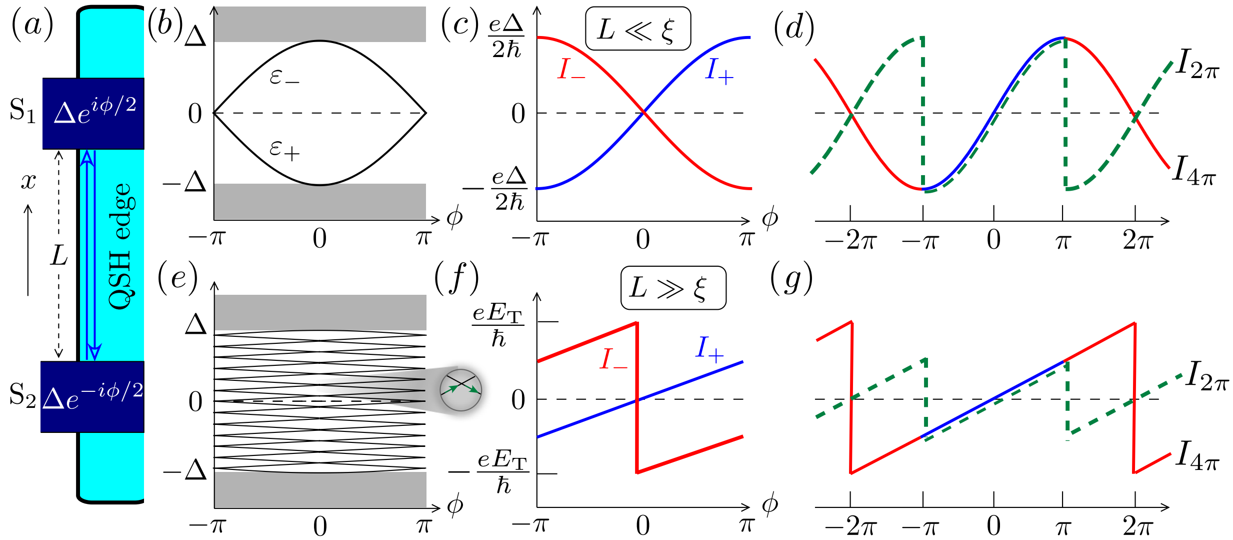

FIG. 1 (color online). Phase-dependent excitation spectrum of a Josephson junction along a quantum spin-Hall (QSH) edge (left panels) and corresponding zero-temperature supercurrent (right panels). The supercurrent $I_{4\pi}$ is 4π-periodic, with two branches I_+ (blue solid), I_- (red solid) distinguished by the ground-state fermion parity and with a parity switch at $\phi = \pm\pi$. The top row shows the short-junction limit of Ref. [7], the bottom row the long-junction limit calculated here. (The jump in I_- at $\phi = 0$ occurs because of the change in slope indicated by the green arrows in the magnified central part of the spectrum.) The 2π-periodic supercurrent $I_{2\pi}$ without parity constraints is also shown (green dashed). The critical current is the same for $I_{4\pi}$ and $I_{2\pi}$ in the short junction, but different by a factor of two in the long junction.

$$I_\pm(\phi) = \pm\frac{e\Delta}{2\hbar}\sin(\phi/2), \qquad |\phi| < \pi. \tag{2}$$

To discuss the fermion-parity anomaly we assume, for definiteness, that the total number $\mathcal{N}$ of electrons in the system is even. (A different choice amounts to a 2π phase shift, or equivalently, an interchange of I_+ and I_-.) The ground-state fermion parity σ is even for $\phi = 0$ and switches to odd when ϕ crosses π. Since $\mathcal{N}$ is fixed, this topological phase transition must be accompanied by a switch between an even and odd number of quasiparticle excitations. At zero temperature, only the two levels $\varepsilon_\pm$ closest to the Fermi level ($\varepsilon = 0$) play a role, and the parity switch of σ means that a quasiparticle is transferred from $\varepsilon_+ < 0$ to $\varepsilon_- > 0$. It cannot relax back from ε_- to ε_+ at fixed parity of $\mathcal{N}$.

The resulting current-phase relationship can be represented by a switch between 2π-periodic branches $I_\pm(\phi)$ (reduced zone scheme), or equivalently as a 4π-periodic function $I_{4\pi}(\phi)$ (extended zone scheme). Both representations are shown in Fig. 1, upper panels. We also include the 2π-periodic current $I_{2\pi}$ that results if the system can relax to its lowest energy state without constraints on the parity of $\mathcal{N}$.

So much for the short-junction limit. An elementary discussion of the long-junction regime (to be made rigorous in just a moment) goes as follows. For $L \gg \xi$ we may assume [22–24] a local linear relation between the current density I and the phase gradient $\phi/L \ll 1/\xi$, of the form $I = \text{const} \times ev\phi/L$. The linear increase of I_- is interrupted at $\phi = 0$ by a discontinuity $\Delta I_- = 2ev/L$. Half of it results from the jump in the slope of the lowest occupied positive energy level $\varepsilon = (\pi - |\phi|)\hbar v/2L$ [green arrows in Fig. 1(e)]. The jump in the slope of the highest occupied negative energy level contributes the other half. In the extended zone scheme, the resulting supercurrent $I_{4\pi}$ is a 4π-periodic sawtooth with a slope $\Delta I_-/4\pi = eE_T/2\pi\hbar$.

The corresponding parity-dependent supercurrents in the reduced zone scheme are

$$I_+ = \frac{eE_T}{2\pi\hbar}\phi, \qquad I_- = \frac{eE_T}{2\pi\hbar}(\phi - 2\pi\,\mathrm{sgn}\phi), \qquad |\phi| < \pi. \tag{3}$$

The 4π-periodic supercurrent $I_{4\pi}$ switches from I_+ to I_- at $\phi = \pi$, while $I_{2\pi}$ remains in the branch I_+ by compensating the switch in ground-state fermion parity σ by a switch in the parity of the electron number $\mathcal{N}$. These are the curves plotted in Fig. 1 (lower panels).

The maximal supercurrent is reached near $\phi = 2\pi$ for $I_{4\pi}$ (with parity constraint) and near $\phi = \pi$ for $I_{2\pi}$ (without parity constraint). There is a factor of two difference in magnitude of these critical currents in a long junction,

$$I_{4\pi,c} = eE_T/\hbar, \qquad I_{2\pi,c} = eE_T/2\hbar. \tag{4}$$

In contrast, for a short junction both are the same (equal to $e\Delta/2\hbar$).

To determine the crossover from the short-junction limit (2) to the long-junction limit (3), including the temperature dependence, we adapt the scattering theory of the

Josephson effect [25] to include the fermion parity constraints. Input is the scattering matrix s_0 of electrons in the normal region and the Andreev reflection matrix r_A at the normal-superconductor interfaces. These take a particularly simple 2×2 form at the quantum spin-Hall edge, but our general formulas are applicable also to *multichannel* topological superconductors.

The parity-dependent partition function is [12–14,26]

$$Z_\pm = \frac{1}{2}\Big(\prod_{\varepsilon>0} e^{\beta\varepsilon/2}\Big)\Big[\prod_{\varepsilon>0}(1+e^{-\beta\varepsilon}) \pm \prod_{\varepsilon>0}(1-e^{-\beta\varepsilon})\Big]$$
$$= \frac{1}{2} Z_0\Big[1 \pm \prod_{\varepsilon>0}\tanh(\beta\varepsilon/2)\Big], \tag{5}$$

with $\beta = 1/k_B T$ and $Z_0 = \prod_{\varepsilon>0} 2\cosh(\beta\varepsilon/2)$ the partition function without parity constraints. From the expression for $Z_\pm$ one can see that the $\pm$ selects terms that contain an even (+) or an odd (−) number of quasiparticle excitation factors $e^{-\beta\varepsilon}$, as is dictated by the ground-state fermion parity. The partition function Z gives the free energy F and hence the supercurrent I [27],

$$I_\pm = \frac{2e}{\hbar}\frac{dF_\pm}{d\phi}, \qquad F_\pm = -\beta^{-1}\ln Z_\pm, \tag{6}$$

$$I_{2\pi} \equiv I_0 = \frac{2e}{\hbar}\frac{dF_0}{d\phi}, \qquad F_0 = -\beta^{-1}\ln Z_0. \tag{7}$$

The density of states $\rho(\varepsilon)$ contains both the discrete spectrum for $|\varepsilon| < \Delta$ (a sum of delta functions at the Andreev levels) and the continuous spectrum for $|\varepsilon| > \Delta$, including also a contribution ρ_S from the superconducting electrodes. Scattering theory gives the expression [25]

$$\rho(\varepsilon) = \mathrm{Im}\frac{d}{d\varepsilon}\nu(\varepsilon + i0^+) + \rho_S(\varepsilon), \tag{8}$$

$$\nu(\varepsilon) = -\pi^{-1}\ln \mathrm{Det}\, X(\varepsilon), \quad X = (1-M)M^{-1/2}, \tag{9}$$

$$M(\varepsilon) = r_A^*(-\varepsilon)s_0^*(-\varepsilon)r_A(\varepsilon)s_0(\varepsilon). \tag{10}$$

The factor $M^{-1/2}$ in the definition of X, as well as the term ρ_S, give a ϕ-independent additive contribution to F_0 without any effect on I_0, but we need to retain these terms here because they do enter into the parity constraint for $I_\pm$.

In the absence of parity constraints, Ref. [28] gives the free energy

$$F_0 = -\beta^{-1}\sum_{p=0}^{\infty}\ln \mathrm{Det}\, X(i\omega_p), \tag{11}$$

as a sum over fermionic Matsubara frequencies $\omega_p = (2p+1)\pi/\beta$. A similar calculation [29] gives the parity dependence in the form

$$F_\sigma = F_0 - \beta^{-1}\ln\frac{1}{2}\Big[1 + \sigma e^{J_S}\sqrt{\mathrm{Det}\, X(0)}$$
$$\times \exp\Big(\sum_{p=1}^{\infty}(-1)^p \ln \mathrm{Det}\, X(i\Omega_p/2)\Big)\Big], \tag{12}$$

$$\sigma = \mathrm{sgn}[\mathrm{Pf}(r_A s_0 - s_0^T r_A^T)(\mathrm{Det}\, i s_0)^{-1/2}]_{\varepsilon=0}, \tag{13}$$

with bosonic Matsubara frequencies $\Omega_p = 2p\pi/\beta$. The ground-state fermion parity σ is given in terms of the Pfaffian of the antisymmetrized scattering matrix, evaluated at the Fermi energy. The sign ambiguity in the square root is resolved by fixing $\sigma = 1$ at $\phi = 0$.

Equation (12) contains a contribution from the superconducting electrodes,

$$J_S = \int_\Delta^\infty d\varepsilon \rho_S(\varepsilon)\ln\tanh(\beta\varepsilon/2), \tag{14}$$

which only plays a role at temperatures $T \gtrsim \Delta/k_B$. The factor e^{J_S} can therefore be replaced by unity in the long-junction regime, when $k_B T \lesssim E_T \ll \Delta$.

We now specify these general formulas for the quantum spin-Hall edge, with the Hamiltonian [30]

$$H_{\mathrm{BdG}} = \begin{pmatrix} vp\sigma_z + U(x) & \Delta^*(x)\sigma_y \\ \Delta(x)\sigma_y & vp\sigma_z - U(x)\end{pmatrix}. \tag{15}$$

The edge runs along the x axis, $p = -i\hbar\partial_x$ is the momentum operator, and the electrostatic potential is $U(x)$ (measured relative to the Fermi level). The pair potential $\Delta(x)$ vanishes in the normal region $|x| < L/2$. In the two superconducting regions we set $\Delta(x) = \Delta e^{\pm i\phi/2}$, with a step at $x = \pm L/2$. This so-called "rigid boundary condition" is justified for a single channel coupled to a bulk superconducting reservoir [10].

A mode-matching calculation gives the scattering matrices

$$s_0 = \begin{pmatrix} 0 & e^{i\chi} \\ e^{i\chi} & 0 \end{pmatrix}, \qquad \chi(\varepsilon) = \chi_0 + \varepsilon/E_T, \tag{16}$$

$$r_A = \begin{pmatrix} \alpha e^{i\phi/2} & 0 \\ 0 & -\alpha e^{-i\phi/2}\end{pmatrix}, \qquad \alpha(\varepsilon) = \sqrt{1 - \frac{\varepsilon^2}{\Delta^2}} + \frac{i\varepsilon}{\Delta},$$

$$\mathrm{Det}\, X(\varepsilon) = 2\cos\phi + \alpha^2 e^{2i\varepsilon/E_T} + \alpha^{-2}e^{-2i\varepsilon/E_T}. \tag{17}$$

We discuss the various terms in these expressions. The electron scattering matrix s_0 is purely off diagonal, because of the absence of backscattering along the quantum spin-Hall edge. The transmission phase χ depends linearly on energy because of the linear dispersion. Electrostatic potential fluctuations contribute only to the energy-independent offset $\chi_0 = -(\hbar v)^{-1}\int_0^L U dx$, which drops out in Eq. (9). The Andreev reflection matrix r_A (from electron to hole) is unitary below the gap. Above the gap there is also propagation into the superconductor, so r_A is subunitary. The same expression (16) for r_A applies at all energies, evaluated at $\varepsilon + i0^+$ to avoid the branch cut of the square root.

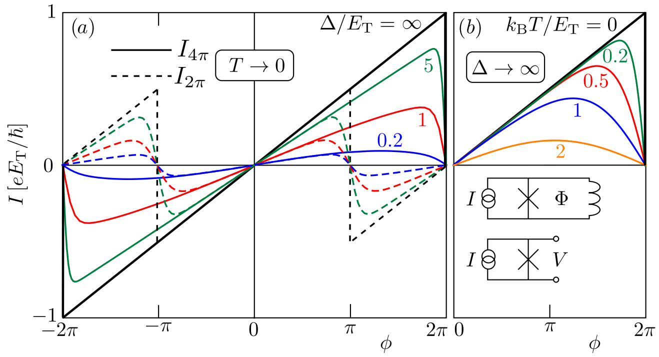

FIG. 2 (color online). Phase dependence of the parity-constrained supercurrent $I_{4\pi}$ (solid curves, in units of $eE_T/\hbar \propto 1/L$), calculated by a numerical evaluation of the Matsubara sums. The left panel shows the crossover from the short-junction to the long-junction regime in the zero-temperature limit (full interval $-2\pi < \phi < 2\pi$). The right panel shows the temperature dependence in the long-junction limit (reduced interval $0 < \phi < 2\pi$). The left panel also shows the supercurrent $I_{2\pi}$ without parity constraints (dashed curves). The insets in the right panel show current-biased superconducting circuits that measure the I-V and I-Φ relationships of a Josephson junction.

Putting all pieces together [29], we obtain the parity-dependent supercurrent for arbitrary ratio Δ/E_T. In the short-junction limit $\Delta/E_T \to 0$ we recover the known result (2), when the energy dependence of the scattering matrix and the phase sensitivity of the continuous spectrum can both be ignored. In the opposite long-junction limit $\Delta/E_T \to \infty$ we find

$$I_{4\pi} = I_0 - \frac{2e}{\hbar\beta}\frac{d}{d\phi}\ln\left[\frac{1}{2} + \cos(\phi/2)e^{\mathcal{S}-\pi/2\beta E_T}\right], \quad (18)$$

$$\mathcal{S} = \sum_{p=1}^{\infty}(-1)^p \ln(1 + 2e^{-\Omega_p/E_T}\cos\phi + e^{-2\Omega_p/E_T}), \quad (19)$$

$$I_{2\pi} \equiv I_0 = \frac{2e}{\hbar\beta}\sin\phi\sum_{p=0}^{\infty}[\cos\phi + \cosh(2\omega_p/E_T)]^{-1}. \quad (20)$$

The plot of the results in Fig. 2 shows that the crossover from a sine to a sawtooth shape occurs early: already for $\Delta = E_T$ (so for $L = \xi$) the maximum of the current-phase relationship is close to $\phi = 2\pi$. The sawtooth shape is preserved with increasing temperature for $k_B T \lesssim \frac{1}{2}E_T$.

These are encouraging results for the experimental accessibility of the long-junction regime. The quantum spin-Hall effect has been observed in HgTe/CdTe quantum wells [31], and more recently in InAs/GaSb quantum wells [32]—where also Andreev reflection from superconducting Nb electrodes was demonstrated [33]. For a typical Fermi velocity of $v \simeq 10^5$ m/s in a semiconductor and superconducting gap $\Delta \simeq 1$ meV in bulk Nb, the coherence length is $\xi = 70$ nm, so the Josephson junction length $L = 0.5$ μm from Ref. [33] is deep in the long-junction regime. Since the long-junction regime is already entered for $L \approx \xi$, this would apply even if the effective superconducting gap is well below the bulk value of Nb. The corresponding Thouless energy is $E_T/k_B = 1.5$ K, so at $T = 100$ mK one should be close to the low-temperature limit.

In the ongoing search for the 4π-periodic Josephson effect the first results have been reported [34] for the ac effect (fractional Shapiro steps [9,15–18]). A dc measurement of the current-flux (I-Φ, $\phi = 2e\Phi/\hbar$) relationship, for times large compared to the time $\tau_{qp} \simeq \mu$s for unpaired quasiparticles to tunnel into the system [35], will measure the 2π periodic $I_{2\pi}$ rather than $I_{4\pi}$. Such a phase-sensitive measurement (Fig. 2, upper inset) would produce the critical current $I_{2\pi,c}$ without any signature of the parity anomaly. In contrast, a phase-*insensitive* measurement of the critical current through the current-voltage (I-V) characteristic (lower inset) will produce $I_{4\pi,c}$ even on time scales $\gg \tau_{qp}$, because the phase of a resistively shunted (overdamped) circuit can adjust to a change in $\mathcal{N}$ on time scales much smaller than τ_{qp}. A change in the parity of $\mathcal{N}$ will be compensated by a 2π phase shift, without a change in critical current [29]. In a short junction, $I_{2\pi,c}$ and $I_{4\pi,c}$ are the same, so this does not help, but in a long junction they differ by up to a factor of two.

In conclusion, we have presented a theory for the 4π-periodic Josephson effect on large scales compared to the superconducting coherence length. A multitude of subgap states, as well as a continuum of states above the gap, contribute to the supercurrent for $L \gg \xi$, but still the parity anomaly responsible for the 4π periodicity persists. In fact, we have found that in a long junction the anomaly manifests itself also in a phase-insensitive way, through a doubling of the critical current. This opens up new possibilities for the detection of this topological effect at the quantum spin-Hall edge [31–33], and possibly also in semiconductor nanowires [34,36–41].

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