Potential Vorticity Formulation of Compressible Magnetohydrodynamics

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Compressible ideal magnetohydrodynamics is formulated in terms of the time evolution of potential vorticity and magnetic flux per unit mass using a compact Lie bracket notation. It is demonstrated that this simplifies analytic solution in at least one very important situation relevant to magnetic fusion experiments. Potentially important implications for analytic and numerical modelling of both laboratory and astrophysical plasmas are also discussed.

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Introduction.—Ideal magnetohydrodynamics (MHD) is a model for magnetized plasma where the collisionality is low, so that dissipative effects can be neglected, yet where the charged particles still interact sufficiently strongly via the electromagnetic field they can be treated as a single fluid. The ideal MHD model is applied to a wide range of laboratory and astrophysical situations, where there are long periods of relative quiescence in which Maxwellian particle distributions can be approached, interrupted by often violent transients. Ideal MHD instabilities are thought to be implicated in the triggering of the sawtooth crash phenomenon in tokamak magnetic fusion experiments and flaring in the solar and stellar context, see textbooks such as [1]. The former is important as it limits the performance of devices ultimately intended to generate nuclear power, and the latter is implicated in the generation of solar magnetic storms which can disrupt terrestrial power grids, navigation, and communication systems. Both these topics are presently the subject of intensive investigation, magnetic fusion as the multibillion dollar ITER tokamak enters the construction phase, whereas multiple satellite missions are collecting data on solar and stellar magnetic fields.

It is often mathematically convenient when employing ideal MHD, to assume that the plasma fluid is *incompressible*, but the reality in the above-mentioned situations is that the plasma density varies by one or more orders of magnitude over the region of interest. This work presents what is believed to be a novel, mathematically convenient formulation of *compressible* MHD.

The equations of ideal MHD as usually formulated are well known and are to be found in many textbooks; see, e.g., Ref. [1], Sec. 4.3. As explained there, the problem admits a variational formulation which is of great utility for practical stability analysis, and a functional Hamiltonian formulation in terms of Lie derivatives [2], of great theoretical importance for understanding stability and evolution. More direct approaches to ideal MHD stability are also now used ([1], Sec. 6), and the results presently to be described are more relevant to the latter school.

The potential vorticity is the ordinary vorticity $\boldsymbol{\omega}$ of the plasma (the curl of the mean flow $\mathbf{U}$ of ions and electrons), divided by the mass density ρ, i.e., $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega}/\rho$. The possibility of combining the equation for the time evolution of vorticity with that for density evolution to give a simple equation for the rate of change of potential vorticity, was first realized for a classical fluid by Helmholtz as described by Ref. [3], Sec. 146, in the mid-19th century. In the mid-20th century, Walén, according to Ref. [4], Sec. 4-2, was the first to realize that a mathematically identical relation governed the evolution of the magnetic flux per unit mass $\tilde{\mathbf{B}} = \mathbf{B}/\rho$ where $\mathbf{B}$ is the magnetic field. For *incompressible* plasma, Arnold and Khesin [5], Sec. I.10.C, combined these results in late-20th century to give an elegant formulation of ideal MHD in terms of Lie brackets of vector fields. The Lie bracket is here the generalization to arbitrary vector fields of the "flux-freezing" operator, i.e., the operator which determines the advection of divergence-free (solenoidal) fields $\mathbf{B}$ and $\boldsymbol{\omega}$ [6], Sec. 3.8. The novelty of the present work is to extend this formalism to *compressible* MHD and explore the implications. In particular, the peculiar, coordinate invariant nature of the Lie bracket makes it easy to generalize solutions to arbitrary geometry in some cases, both analytically and numerically.

The next section contains a detailed mathematical derivation of the key formula. A discussion of the implications for analytic and numerical solution follows, and finally some important possible applications are summarized.

Mathematics.—In terms of the operators of classical vector mechanics, the Lie derivative of a vector can be defined as

$$\mathcal{L}_{\mathbf{u}}(\mathbf{v}) = \nabla \times (\mathbf{u} \times \mathbf{v}) - \mathbf{u}\nabla \cdot \mathbf{v} + \mathbf{v}\nabla \cdot \mathbf{u}, \qquad (1)$$

which will help explain the equivalence with the vector advection operator, the first term on the right. Indeed, Walén's result for magnetic induction in a perfectly conducting medium is

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = \mathcal{L}_{\mathbf{U}}(\tilde{\mathbf{B}}). \qquad (2)$$

Introducing component notation for vectors in general nonorthogonal coordinate systems, as described in many textbooks, e.g., Ref. [7], it turns out that the Jacobians thereby introduced (of the coordinate transformation from Cartesians), cancel among the terms in Eq. (1), so that

$$\mathcal{L}_{\mathbf{u}}(\mathbf{v})^i = v^k \frac{\partial u^i}{\partial x^k} - u^k \frac{\partial v^i}{\partial x^k}, \tag{3}$$

where u^k, v^k are the contravariant components of the vectors $\mathbf{u}$, $\mathbf{v}$, respectively, and the summation convention is implied. It follows that

$$\mathcal{L}_{\mathbf{u}}(\mathbf{v}) = -\mathcal{L}_{\mathbf{v}}(\mathbf{u}) = -[\mathbf{u}, \mathbf{v}], \tag{4}$$

where [.,.] denotes the Lie bracket of Schutz [8].

It will now be proved that the equation for the evolution of potential vorticity in compressible ideal MHD may be written

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} = \mathcal{L}_{\mathbf{U}}(\tilde{\boldsymbol{\omega}}) - \mathcal{L}_{\tilde{\mathbf{B}}}(\tilde{\mathbf{J}}), \tag{5}$$

where the potential current $\tilde{\mathbf{J}} = \nabla \times \mathbf{B}/\rho$. The customary vorticity equation in ideal MHD is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{U} \times \boldsymbol{\omega}) + \frac{\nabla \rho \times \nabla p}{\rho^2} + \nabla \times \left(\frac{\mathbf{J} \times \mathbf{B}}{\rho}\right), \tag{6}$$

where vorticity $\boldsymbol{\omega} = \rho\tilde{\boldsymbol{\omega}} = \nabla \times \mathbf{U}$, and current $\mathbf{J} = \rho\tilde{\mathbf{J}} = \nabla \times \mathbf{B} = \nabla \times (\rho\tilde{\mathbf{B}})$. When proceeding further, it is convenient and often physically justifiable, by a barotropic or isentropic assumption, to neglect the term in the pressure p, and if not, the resulting additional term is easily representable in general geometry.

It follows that to establish the equivalence of Eqs. (5) and (6), it is necessary to show that $\boldsymbol{\Delta} = \mathbf{0}$, where

$$\boldsymbol{\Delta} = \frac{1}{\rho} \nabla \times \left(\frac{\mathbf{B} \times \mathbf{J}}{\rho}\right) - \mathcal{L}_{\tilde{\mathbf{B}}}(\tilde{\mathbf{J}}). \tag{7}$$

Now, Eq. (7) is a vector equation, so validity in any coordinate frame implies validity in all; hence, it is sufficient to establish the result in Cartesian coordinates, where

$$\boldsymbol{\Delta} = \frac{1}{\rho} \nabla \times (\rho\tilde{\mathbf{B}} \times \tilde{\mathbf{J}}) + \tilde{\mathbf{B}} \cdot \nabla\tilde{\mathbf{J}} - \tilde{\mathbf{J}} \cdot \nabla\tilde{\mathbf{B}}. \tag{8}$$

The curl term may be expanded using the identity

$$\frac{1}{\rho} \nabla \times (\rho \mathbf{v}) = \mathbf{R} \times \mathbf{v} + \nabla \times \mathbf{v}, \tag{9}$$

where $\mathbf{R} = \nabla\rho/\rho$. Setting $\mathbf{v} = \tilde{\mathbf{B}} \times \tilde{\mathbf{J}}$, and expanding the resulting curl-cross operation, there is cancellation of the two terms from the Lie derivative, leaving

$$\boldsymbol{\Delta} = \tilde{\mathbf{B}}\nabla \cdot \tilde{\mathbf{J}} - \tilde{\mathbf{J}}\nabla \cdot \tilde{\mathbf{B}} + \mathbf{R} \times (\tilde{\mathbf{B}} \times \tilde{\mathbf{J}}). \tag{10}$$

Since $\nabla \cdot \mathbf{J} = 0$, it follows that

$$\nabla \cdot \tilde{\mathbf{J}} = -\mathbf{R} \cdot \tilde{\mathbf{J}}, \tag{11}$$

and likewise since $\nabla \cdot \mathbf{B} = 0$,

$$\nabla \cdot \tilde{\mathbf{B}} = -\mathbf{R} \cdot \tilde{\mathbf{B}}. \tag{12}$$

Substituting Eqs. (11) and (12) into Eq. (10), and expanding the last term as dot products, shows that, as required $\boldsymbol{\Delta} = \mathbf{0}$.

The set of evolution equations is completed by mass conservation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{U}). \tag{13}$$

This does not involve a vector Lie derivative, but, using the standard expression for the divergence operator in general curvilinear coordinates, it may be written

$$\frac{\partial \rho}{\partial t} = -\frac{1}{\sqrt{g}} \frac{\partial(\rho\sqrt{g}U^k)}{\partial x^k}, \tag{14}$$

where $\sqrt{g}$ is the Jacobian and the g_{ik} is the metric tensor, which upon introducing $\tilde{\rho} = \rho\sqrt{g}$ may be written

$$\frac{\partial \tilde{\rho}}{\partial t} = -\frac{\partial(\tilde{\rho}U^k)}{\partial x^k}, \tag{15}$$

provided that $\sqrt{g}$ does not change with time. Like the neglect of the pressure term above, this latter inessential assumption is often physically reasonable.

Unfortunately, the ideal MHD equations are here completed by the two definitions of potential vorticity and potential current, which *do* explicitly contain metric information, viz.

$$\tilde{\rho}\tilde{\omega}^i = e^{ikl} \frac{\partial(g_{ln}U^n)}{\partial x^k}, \tag{16}$$

and

$$\tilde{\rho}\tilde{J}^i = e^{ikl} \frac{\partial}{\partial x^k}\left(\frac{g_{ln}}{\sqrt{g}}\tilde{\rho}\tilde{B}^n\right). \tag{17}$$

In the above, $e^{ikl} = e_{ikl}$ is the alternating symbol, taking values 1, −1, or 0, depending on whether (ikl) is an even, odd, or nonpermutation of (123). Finally, note that Eqs. (2) and (15) together ensure that $\nabla \cdot \mathbf{B} = 0$, only if initially

$$\frac{\partial(\tilde{\rho}\tilde{B}^k)}{\partial x^k} = 0. \tag{18}$$

Solving the new system.—The new model system for ideal barotropic compressible MHD evolution consists of Eq. (2), (5), and (15)–(17). The simplification of the first three has been gained at the expense of complicating the last two "static" relations. Nonetheless, evolution equations are harder to treat numerically, because any errors in the discretization tend to combine over time. Moreover, observing that equations of the form Eq. (2) have the solution in Lagrangian coordinates $\boldsymbol{\xi}$

$$\tilde{B}^i(\boldsymbol{\xi}(t), t) = \frac{\partial \xi^i}{\partial \xi_0^j} \tilde{B}^j(\boldsymbol{\xi}_0, 0), \quad \text{where } \frac{d\boldsymbol{\xi}}{dt} = \mathbf{U}, \quad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 \tag{19}$$

offers the possibility of using radically different, ephemeral particle-in-cell (EPIC) schemes [9] for the time update. To this end, the term nonlinear in the magnetic field is better expressed as $+\mathcal{L}_{\tilde{\mathbf{J}}}(\tilde{\mathbf{B}})$, so that the change in $\tilde{\mathbf{B}}$ calculated by EPIC under the "flow" $d\boldsymbol{\xi}/dt = \tilde{\mathbf{J}}$ may be used in a split update of $\tilde{\boldsymbol{\omega}}$. By employing EPIC or similar algorithms it is also possible to envisage schemes where the mesh remains logically fixed while the metric changes with each field update.

Using more conventional finite difference or finite element schemes, it will be evident that problems solved in Cartesian geometry will test all aspects of the coding of the evolutionary equations. Thus, there is considerable computational advantage to be gained here also. There is obviously the concern that the magnetic field computed may not be accurately solenoidal, but this is an issue for many other discretizations also. The main difficulty is in the inversion of Eq. (16) to give the velocity field $\mathbf{U}$ corresponding to a freshly evolved potential vorticity [since $\tilde{\mathbf{B}}$ itself is evolved, Eq. (17) does not need to be inverted]. However, this inversion, together with the computation of the irrotational part of $\mathbf{U}$, is a classical hydrodynamical problem, and a variety of strategies may be found in the literature. On present machine architectures, introducing the vector potential for velocity then solving the coupled system Eqs. (15) and (16) by a pseudotimestepping algorithm is probably to be preferred. Similar numerical solution strategies were successfully employed in electromagnetics by the current author and collaborators [10,11]. Vorticity formulations are common in plasma modeling as they are helpful in several physically relevant limits, and in particular, a vorticity formulation has already been used successfully in nonlinear, compressible MHD [12].

Turning to analytic results, first consider MHD equilibrium solutions with no time dependence and $\mathbf{U} = \mathbf{0}$, implying $\mathcal{L}_{\tilde{\mathbf{B}}}(\tilde{\mathbf{J}}) = 0$. In the case of force-free fields, meaning $\mathbf{J} \propto \mathbf{B}$, substituting $\tilde{\mathbf{J}} = \lambda\tilde{\mathbf{B}}$ in the Lie derivative in component form show this is a solution provided $\mathbf{B}\cdot\nabla\lambda = 0$, i.e., exactly the same constraint on λ that follows from the solenoidal constraint on $\mathbf{B}$ and $\mathbf{J}$ when seeking the solution $\mathbf{J} = \lambda\mathbf{B}$. Hydromagnetic force-free solutions, with the additional constraint that $\mathbf{U} = \lambda_2\mathbf{B}$, now cease to exist, however, because $\mathbf{U}$ is not solenoidal unless the flow is incompressible.

Moving now to time dependent solutions, interest attaches to the "flux compression" solution [13], Sec. 4.6, which is postulated on purely kinematic grounds [i.e., from Eq. (2)] and which may be written

$$\mathbf{B} = c_B(0, 0, \rho(x, y, t)) \tag{20}$$

for a compressible flow $\mathbf{U}$ with density ρ provided that $\mathbf{U} = (U_x(x, y, t), U_y(x, y, t), 0)$. Here, c_B is an arbitrary constant and (x, y, z) are the usual Cartesian coordinates. This solution is of practical importance for fusion experiments, where external magnets are used to generate a time dependent flux designed so as to compress plasma "frozen" to it. It is easy to establish that if $\mathbf{B} = c_B\rho\hat{\mathbf{z}}$, then

$$\mathbf{J}\times\mathbf{B}/c_B^2 = (\nabla\rho\times\hat{\mathbf{z}})\times\rho\hat{\mathbf{z}} = -\nabla(\rho^2/2); \tag{21}$$

i.e., the associated Lorentz force does not generate vorticity and the more complicated dynamics which might follow.

The simple form of the new evolution equations enables a generalization of the flux-compression solution to general curvilinear coordinates. It is important to emphasize that the following is not simply reexpressing $\mathbf{B} = \rho\hat{\mathbf{z}}$ in different coordinate systems, nor is there a loss of generality in choosing units for density such that $c_B = 1$. The obvious generalization is to take $\tilde{B}^3 = 1(\tilde{B}^1 = \tilde{B}^2 = 0)$, implying a 2D density to ensure a solenoidal $\mathbf{B}$, since Eq. (18) requires $\partial\tilde{\rho}/\partial x^3 = 0$. The next step is to ensure that $\mathcal{L}_{\tilde{\mathbf{B}}}(\tilde{\mathbf{J}}) = 0$, which as may be seen using the coordinate form Eq. (3), simply requires $\partial\tilde{J}^j/\partial x^3 = 0$. Similarly $\mathcal{L}_{\mathbf{U}}(\tilde{\mathbf{B}}) = 0$ may be satisfied by a flow with $\partial U^j/\partial x^3 = 0$ (note that $U^3 \neq 0$ is therefore allowed). Last, from the "static" relations, it will be seen that a solution with $\tilde{J}^j$ independent of x^3 is possible provided $\partial g_{ik}/\partial x^3 = 0$. The general geometry analogue of Eq. (21) may now be computed as

$$\frac{\mathbf{J}\times\mathbf{B}^i}{\rho} = -\frac{\partial}{\partial x^i}(c_B^2 g_{33}\rho), \tag{22}$$

confirming that the Lorentz force per unit mass is irrotational and therefore does not produce or reduce vorticity.

Thus, suppose that the equations of 2D compressible hydrodynamics subject to an additional force Eq. (22) have the time dependent solution

$$\begin{aligned} U^i &= (U^1(x^1, x^2, t), U^2(x^1, x^2, t), U^3(x^1, x^2, t)),\\ \rho &= \rho(x^1, x^2, t), \end{aligned} \tag{23}$$

then the preceding paragraph shows that no vorticity is generated by the time dependent field $\mathbf{B}$ given by $B^1 = B^2 = 0$, $B^3 = c_B\rho$, in any coordinate system where $\partial g_{ik}/\partial x^3 = 0$. Or equivalently, in the language of differential geometry [8], Sec. 3.11, if $\tilde{\mathbf{B}}$ is a Killing vector, there is a flux-compression solution.

The Killing vectors for 3D are textbook [6] and there are three main possibilities, corresponding to the cases where everywhere $\mathbf{B}$ is parallel to a coordinate axis, either (i) any Cartesian axis, or (ii) polar angle direction in cylindrical coordinates, or (iii) a helical axis in a system of helical coordinates with constant pitch.

To answer possible questions about the existence of 2D solutions subject to a force proportional to density as in Eq. (22), observe that the pressure force is $-\nabla p/\rho$. Hence, if $p \propto \rho^\gamma$ where the polytropic index $\gamma = 2$, at least in Cartesians this force takes the same form as Lorentz. The mathematical theory establishing the existence of (irrotational) flow applies for arbitrary γ [14], Sec. 7.4.

Other possibilities for new analytic solutions outside of $\tilde{B}^3 = 1$ are opened up when it is realized that Clebsch variables [7], Sec. 5, may be used to represent a solenoidal vector field as a single contravariant component. Alternatively, the vector potential may be introduced, leading to an interesting calculus involving $\mathbf{R}$, consistent with the fact that exponentially varying density profiles (implying constant $\mathbf{R}$) are often studied analytically.

Applications.—For discharge stability in fusion physics, flux-compression solutions have not historically been relevant because the associated Lorentz force is outward from a region of high density. However, advanced tokamak equilibria may have regions of density inversion, and high rates of velocity shear may contribute to confinement via the $\frac{1}{2}\nabla U^2$ Bernouilli pressure term. Hence the above, new analytic flux-compression solutions may represent a nonlinear development of interchange modes [15], Sec. 12.1.2, in regions where either the confining field has approximately constant twist corresponding to Killing vector (iii), or is approximately toroidal, vector (ii). These solutions, with or without vorticity, would seem to represent efficient and rapid means whereby mass and hence heat might escape from a discharge, so might be implicated in situations where there is rapid transient cooling, such as the sawtooth crash in the center of the tokamak discharge (constant field-line twist), and edge localized modes in divertor discharges (nearly toroidal field lines). The preceding section has also indicated that the new formalism could be used efficiently to simulate ideal MHD evolution of discharges in generalized coordinates, say defined by an arbitrary MHD equilibrium.

In astrophysics, observed magnetic fields usually exhibit a significant degree of disorder, so it is unclear how important the new flux-compression solutions might be, as they rely on at least a degree of coordinate invariance. It is speculated that, in stars with a strong internal toroidal field (such as the Sun is believed to possess), the rotationally symmetric solution corresponding to Killing vector (ii) might help model the convection pattern, accounting for the largely latitudinal variation of the solar differential rotation. Regardless, it should be helpful that, in the new equations, the field geometry appears only in the state equations. It will, for example, be simpler to generate more realistic solutions from symmetric ones by varying g_{ik} starting with the unit tensor. This could be useful, say, for modeling sunspot penumbras both analytically and computationally, since there the magnetic field is predominantly directed radially outwards in the horizontal direction.

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