

Implications of Relativity on Nonrelativistic Goldstone Theorems: Gapped Excitations at Finite Charge Density

Alberto Nicolis¹ and Federico Piazza²

¹*Physics Department and Institute for Strings, Cosmology, and Astroparticle Physics, Columbia University, New York, New York 10027, USA*

²*Paris Center for Cosmological Physics and Laboratoire APC, 10, rue Alice Domon et Léonie Duquet, 75205 Paris, France*

(Received 16 April 2012; published 2 January 2013; corrected 7 January 2013)

We adapt the Goldstone theorem to study spontaneous symmetry breaking in relativistic theories at finite charge density. It is customary to treat systems at finite density via nonrelativistic Hamiltonians. Here, we highlight the importance of the underlying relativistic dynamics. This leads to seemingly new results whenever the charge in question is spontaneously broken *and* does not commute with other broken charges. We find that the latter interpolate *gapped* excitations. In contrast, all existing versions of the Goldstone theorem predict the existence of gapless modes. We derive exact nonperturbative expressions for their gaps, in terms of the chemical potential and of the symmetry algebra.

DOI: [10.1103/PhysRevLett.110.011602](https://doi.org/10.1103/PhysRevLett.110.011602)

PACS numbers: 11.30.Qc, 11.10.Wx, 11.30.Cp, 11.30.Fs

Preliminary considerations.—Nonrelativistic Goldstone theorems [1–6] display interesting twists compared to the standard relativistic one [7,8]. First, the nonrelativistic version is notoriously less powerful. For instance, it guarantees the existence of gapless *zero-momentum* excitations but says nothing about their properties, such as, e.g., stability, at finite momenta. A physically relevant example is phonons in superfluid helium-4, which are unstable with a decay rate $\Gamma \sim k^5$ —that is, they are not really eigenstates of the Hamiltonian. Second, as far as counting is concerned, a classic result by Nielsen and Chadha [2] states that, for nonrelativistic systems, the number of gapless Goldstone excitations equals the number of spontaneously broken generators, provided one counts the Goldstone excitations with an *even* dispersion law (e.g., $\omega \sim k^2$) *twice*.

However, to the best of our knowledge, all fundamental interactions are described by relativistic field equations. This means that, for the so-called nonrelativistic systems in the real world, Lorentz invariance is broken only *spontaneously*, i.e., by the state of the system, rather than at the level of the dynamics. It is thus tempting to ask whether the very constrained framework of relativistic field theories can give nontrivial insights into physical systems that are effectively nonrelativistic—and in particular, whether it can be used to sharpen or correct the nonrelativistic versions of the Goldstone theorem.

An immediate reaction to this idea is that in no way can relativistic effects be relevant for systems that, like condensed matter systems in the lab, have a very nonrelativistic equation of state, a very small speed of sound compared to that of light, and so on. But there is more to relativity than just the so-called “relativistic effects,” which are weighed by $(v/c)^2$. First, there is the statement of relativity itself—that all inertial frames are equivalent—which is valid, and powerful, even in the $c \rightarrow \infty$ limit,

corresponding formally to the Galilean limit of Lorentz invariance. Then, there are properties of relativistic field theories that are not directly statements of symmetry but that are nevertheless crucial for the consistency of the theory. For instance, we will see below that the vanishing of commutators for spacelike separated local operators can be used to remove an assumption of the Nielsen-Chadha theorem. Finally—and this will be the useful aspect for our purposes—the fundamental relativistic viewpoint offers an unambiguous starting point to analyze the pattern of spontaneous symmetry breaking for spacetime symmetries and internal ones.

To clarify this last statement with an example, let us consider directly the system we want to focus on for the rest of Letter: a relativistic theory with Hamiltonian H and a group of internal symmetries at finite density for one of the corresponding charges, Q . The ground state $|\mu\rangle$ (i.e., the state of minimal energy for a given average charge density) can be found by the method of Lagrange multipliers as the state minimizing the modified Hamiltonian $\tilde{H} = H - \mu Q$, where μ is the chemical potential [9],

$$\tilde{H}|\mu\rangle = (H - \mu Q)|\mu\rangle = 0. \quad (1)$$

It is standard practice to use the nonrelativistic Hamiltonian \tilde{H} to study the system at finite density [10]. However, it is clear from the outset that we are in the presence of a *spontaneous*—rather than explicit—breaking of Lorentz symmetry. Introducing \tilde{H} is purely a mathematical tool to find a state with the desired properties. The Hamiltonian of the system is still H . The Heisenberg-picture operators evolve in time as dictated by H . In order to better understand the role of \tilde{H} , consider the case in which the (internal) symmetry generated by Q is *also* spontaneously broken, i.e.,

$$\langle\mu|[Q, A(x)]|\mu\rangle \neq 0 \quad (2)$$

for some order parameter $A(x)$. Then, $|\mu\rangle$ cannot be an eigenstate of Q because this would be inconsistent with (2). But, since $|\mu\rangle$ obeys Eq. (1), it cannot be an eigenstate of H either. We conclude that, at finite density for Q , if Q is spontaneously broken, so is H [11]. This means that, if Q is broken, we cannot classify the states of the system—including our ground state $|\mu\rangle$ —as eigenstates of the fundamental Hamiltonian H . The best we can do is try to diagonalize the unbroken combination $\tilde{H} = H - \mu Q$. $|\mu\rangle$ is the eigenstate with the lowest eigenvalue. The excitations of the system—including the Goldstone bosons—will correspond to higher eigenstates.

Since the Nielsen-Chadha theorem implicitly assumes that Heisenberg-picture operators evolve in time with the same (nonrelativistic) Hamiltonian that is minimized by the ground state, we conclude that that theorem *does not apply* to systems that are “nonrelativistic” because of a finite density for a charge that is spontaneously broken. Indeed, for such systems, it is not even clear how one would study the spontaneous breaking of Q in a setup that is nonrelativistic from the start. There are systems of this sort where Q is broken “before” (i.e., at higher energy scales) the Lorentz invariance is broken: Any relativistic theory with ordinary spontaneous symmetry breaking in its Poincaré-invariant vacuum can be put in a state of arbitrarily low density for the broken charge [11].

When there are local operators obeying (2), an alternative viewpoint suggests itself. By Eq. (1), the action of the Hamiltonian on $|\mu\rangle$ is proportional to that of the symmetry generator. We are thus in the presence of a state that evolves in time along a symmetry direction, at “speed” μ . Any field $\phi_j(x)$ that transforms nontrivially under the symmetry can then feature a spacially homogeneous, time-dependent expectation value, obeying

$$\frac{d}{dt}\langle\phi_j\rangle = \mu\langle\delta\phi_j\rangle, \quad (3)$$

where $\phi_j \rightarrow \phi_j + \delta\phi_j$ is the action of the symmetry in field space. In Ref. [11], we dubbed this situation “spontaneous symmetry probing” (SSP)—there, we were using c in place of μ . This viewpoint is particularly useful in the semiclassical limit, where we can think of time evolution in terms of classical trajectories in field space. We refer the reader to Ref. [11] for details.

Assumptions and formalism.—We now want to study the low-energy spectrum of the system at finite density by considering the Goldstone states associated with Q and with other broken generators. Let us assume that the theory enjoys a Lie group of internal symmetries, with generators Q_1, Q_2, \dots, Q_N . Without loss of generality, we can set the first generator to be our Q , $Q = Q_1$. The remarks we made above lead to the following hypotheses: (a) The Heisenberg-picture currents evolve in time as dictated by the original Hamiltonian, i.e., $J_a^\mu(t, \vec{x}) = e^{i(Ht - \vec{P}\cdot\vec{x})} J_a^\mu(0) e^{-i(Ht - \vec{P}\cdot\vec{x})}$, where $a = 1, \dots, N$ and \vec{P} is the total momentum operator;

(b) the state $|\mu\rangle$ is the ground state of $\tilde{H} = H - \mu Q$. The crucially different role played by H and \tilde{H} is the origin of the discrepancy between our results and the existing literature on nonrelativistic Goldstone theorems, e.g., Refs. [2–5].

Consider then the case in which the first n of the Q_a ’s—including Q_1 —are spontaneously broken. By definition, for each spontaneously broken Q_a , there must exist a local operator $A_I(x)$ —an “order parameter”—that makes the matrix element

$$\kappa_{aI} \equiv \langle\mu|[Q_a(t), A_I(0)]|\mu\rangle \quad (4)$$

nonzero. The index $I = 1, \dots, m \leq n$ in general runs over fewer values than the number of broken generators simply because the same $A_I(x)$ typically serves as an order parameter for two or more symmetries.

In the matrix element above, $A_I(x)$ is evaluated at the (spacetime) origin. From now on, to simplify the notation, whenever a local operator is evaluated at the origin, we will drop its argument, $\mathcal{O}(0) \rightarrow \mathcal{O}$. Q_a is formally evaluated—in the Heisenberg picture—at time t . But, since Q_a commutes with the Hamiltonian, it is constant in time, and so is the κ_{aI} matrix element. To convince oneself that this is true even though spontaneously broken charges are not completely well-defined operators, one can use the local conservation of the current:

$$\int d^3x \langle\mu|[J_a^0(\vec{x}, t), A_I]|\mu\rangle + \int d^3x \langle\mu|[\partial_i J_a^i(\vec{x}, t), A_I]|\mu\rangle = 0.$$

The first term is the time derivative of our κ_{aI} , while the second is a boundary term that only receives contributions from spacial infinity. Since, for relativistic quantum field theories, the commutator of spacelike separated local operators vanishes—and we are breaking Lorentz symmetry only spontaneously—such a term is guaranteed to vanish [12]. We conclude that κ_{aI} is constant in time.

We now use assumptions (a) and (b) above to “pull out” of J_a^0 its \vec{x} and t dependence:

$$\begin{aligned} \kappa_{aI} &= \int d^3x \langle\mu|J_a^0(\vec{x}, t)A_I|\mu\rangle - \text{c.c.} \\ &= \int d^3x \langle\mu|e^{i\mu Q}J_a^0e^{-i(Ht - \vec{P}\cdot\vec{x})}A_I|\mu\rangle - \text{c.c.}, \end{aligned} \quad (5)$$

where we used that spacial translations are *not* spontaneously broken, $\vec{P}|\mu\rangle = 0$, and we are assuming that both J_a^0 and A_I are Hermitian operators. Inside the matrix element, we now insert a complete set of intermediate momentum eigenstates $|n, \vec{p}\rangle$, where n labels other quantities that characterize these states. Schematically,

$$\begin{aligned} &\langle\mu|e^{i\mu Q}J_a^0e^{-i(Ht - \vec{P}\cdot\vec{x})}A_I|\mu\rangle \\ &= \sum_{n, \vec{p}} e^{i\vec{p}\cdot\vec{x}} \langle\mu|e^{i\mu Q}J_a^0e^{-i\mu Q}e^{-i\tilde{H}t}|n, \vec{p}\rangle \langle n, \vec{p}|A_I|\mu\rangle, \end{aligned} \quad (6)$$

where we rewrote H in terms of \tilde{H} and Q . The Hamiltonian \tilde{H} commutes with the spatial momentum because H and

Q do. We can then choose the $|n, \vec{p}\rangle$ states to be eigenstates of \tilde{H} as well, with eigenvalues $E_n(\vec{p})$. The integral in d^3x projects onto the zero-momentum states, and we are left with

$$\kappa_{aI} = \sum_n e^{-iE_n(0)t} \langle \mu | e^{i\mu Q t} J_a^0 e^{-i\mu Q t} | n, 0 \rangle \langle n, 0 | A_I | \mu \rangle - \text{c.c.} \quad (7)$$

The constancy in time of the above expression gives important information about the spectrum of the theory in the limit of zero spatial momentum. There are two distinct cases to consider, depending on whether Q_a commutes with $Q \equiv Q_1$ or not. For brevity, let us call these two classes of generators C and NC, short for “commuting” and “noncommuting.” We will show that C generators interpolate states $|n, \vec{p}\rangle$ that have zero energy in the limit of zero momentum, $E_n(0) = 0$. Vice versa, NC generators interpolate states that are *gapped*, in the sense that $E_n(0) \neq 0$.

C generators: Gapless modes.—Equation (7) should be compared with Eq. (6) of Nielsen and Chadha’s paper [2]. They find that the quantity that is constant in time is, in our notation,

$$\kappa_{aI} = \sum_n e^{-iE_n(0)t} \langle \mu | J_a^0 | n, 0 \rangle \langle n, 0 | A_I | \mu \rangle - \text{c.c.} \quad (8)$$

The implicit assumption leading to (8) is that all currents evolve in time with the nonrelativistic Hamiltonian of which $|\mu\rangle$ is the ground state (\tilde{H}). This assumption is violated by our system. However, our Eq. (7) does reduce to (8) whenever Q_a commutes with Q , since in this case $e^{i\mu Q t} J_a^0 e^{-i\mu Q t} = J_a^0$ [13].

Note that, for $|n, 0\rangle = |\mu\rangle$, the combination (8) just gives zero because J_a^0 and A_I are Hermitian operators. In order for κ_{aI} to be time independent *and* different from zero, there must exist a Goldstone state $|\pi, \vec{p}\rangle$ other than $|\mu\rangle$ whose energy goes to zero in the zero-momentum limit, $E_\pi(0) = 0$. Moreover, the matrix elements $\langle \mu | J_a^0 | \pi, \vec{p} \rangle$ and $\langle \pi, \vec{p} | A_I | \mu \rangle$ should be nonzero for such a state: Both the broken current and the order parameter A_I have to interpolate the Goldstone excitation.

Beyond this basic argument, the detailed analysis of the number and nature of gapless Goldstone bosons follows closely that of Ref. [2] and features all the subtleties considered therein (see also Refs. [3,5,6] for more recent refinements). We have nothing to add to the existing analyses, other than to emphasize that they apply here to broken generators of the *C* type only. Notice that, among the C generators, we have Q itself. We devoted the bulk of Ref. [11] to studying the physical properties of the associated Goldstone boson.

NC generators: The gap.—Let us now consider the case where Q_a does not commute with $Q = Q_1$. For our purposes, it is useful to write the commutation relations in hybrid form with charges and currents (Once the algebra for the charges is given, the charge-current commutators

are uniquely determined up to possible contact terms that vanish at zero momentum, i.e., total spacial derivatives. Since we took the $\vec{p} \rightarrow 0$ limit, such possible extra terms will not matter for us.)

$$[Q_a, J_b^0(x)] = if_{ab}^c J_c^0(x), \quad (9)$$

where f_{ab}^c are the group’s structure constants, which are real for any Lie group. We can now go back to Eq. (7). After expanding the exponentials on both sides of the current, using recursively (9) to eliminate all Q ’s and reexponentiating the result, we find

$$e^{i\mu Q t} J_a^0 e^{-i\mu Q t} = (e^{-f_1 \mu t})_a^b J_b^0, \quad (10)$$

where f_1 is a matrix with entries f_{1a}^b . That is, if_1 is the adjoint representation of Q_1 , $(Q_1^A)_a^b = if_{1a}^b$. Equation (10) above is nothing but the usual statement—applied to the currents—that the generators of a group live in the adjoint representation of that group.

The exponential acting on the current is now a finite dimensional matrix that “mixes” in a time-dependent fashion the different currents of the group:

$$\kappa_{aI} = \sum_n e^{-iE_n(0)t} (e^{-f_1 \mu t})_a^b \langle \mu | J_b^0 | n, 0 \rangle \langle n, 0 | A_I | \mu \rangle - \text{c.c.} \quad (11)$$

We will now assume—as usual—that the symmetry group under consideration is the direct product of simple compact Lie groups [SU(n), SO(n), etc.] and of $U(1)$ factors. In this case, the structure constants f_{ab}^c can be taken to be totally antisymmetric—see, e.g., Ref. [8]. Since f_{1a}^b is real and antisymmetric, its eigenvalues are either zero or pure imaginary. The imaginary eigenvalues come in pairs ($+iq_a, -iq_a$), with corresponding Hermitian-conjugate pairs of eigenvectors, defining a (non-Hermitian) basis of generators in which

$$f_{1a}^b = \text{iddiag}(0, \dots, 0, q_1, -q_1, q_2, -q_2, \dots). \quad (12)$$

By acting separately on each $\pm q_a$ block, it is straightforward to define instead a *Hermitian* basis, in which f_{1a}^b is real and block diagonal, with 2×2 blocks of the form

$$\begin{pmatrix} 0 & +q_a \\ -q_a & 0 \end{pmatrix}. \quad (13)$$

Let us assume that we started with Eq. (5) directly in this Hermitian basis where f_{1a}^b is block diagonal, and let us restrict our analysis to the spontaneously broken generators that yield nonvanishing κ_{aI} . If J_a^0 corresponds to a vanishing eigenvalue of $Q_1^A = if_1$ —i.e., if it commutes with Q_1 —then we go back to the C-generator case and we conclude that J_a^0 interpolates gapless particles. If, on the other hand, J_a^0 is either of the paired currents acted upon by a 2×2 block of the form (13), then the exponential in (11) mixes its matrix elements with those of its companion, with frequency μq_a . In this case, the time independence of κ_{aI}

implies a nonzero value for $E_n(0)$. This is most easily seen by using a complex notation: J_a^0 can be expressed as $J_a^0 = \hat{J}_a^0 + \hat{J}_a^{0\dagger}$ —where \hat{J}_a^0 and $\hat{J}_a^{0\dagger}$ are a Hermitian-conjugate pair of non-Hermitian generators that diagonalize f_1 , as in (12)—and Eq. (11) simply becomes

$$\begin{aligned} \kappa_{aI} = & \sum_n e^{-i[E_n(0) - \mu q_a]t} \langle \mu | \hat{J}_a^0 | n, 0 \rangle \langle n, 0 | A_I | \mu \rangle \\ & + e^{-i[E_n(0) + \mu q_a]t} \langle \mu | \hat{J}_a^{0\dagger} | n, 0 \rangle \langle n, 0 | A_I | \mu \rangle - \text{c.c.} \end{aligned} \quad (14)$$

Notice that we are not implicitly summing over a . Without loss of generality, we will assume that μq_a is positive and $-\mu q_a$ negative.

By assumption, our $|\mu\rangle$ state is the ground state of the unbroken \tilde{H} Hamiltonian. Therefore, there cannot be states with negative $E_n(\vec{p})$. So, for the combination (14) to be constant in time and different from zero, two conditions have to be met: (i) There exists a state $|\pi_a, \vec{p}\rangle$ whose energy in the zero-momentum limit is

$$E_a(0) = \mu q_a. \quad (15)$$

This is our main result: the gap for the Goldstone excitations associated with the broken NC generators. (ii) Both \hat{J}_a^0 and A_I are interpolating fields for such a state, in the sense that $\langle \mu | \hat{J}_a^0 | \pi_a, \vec{p} \rangle \neq 0$ and $\langle \mu | A_I | \pi_a, \vec{p} \rangle \neq 0$. (Recall that A_I is Hermitian.)

Notice that we have *one* gapped Goldstone mode for each *pair* of broken NC generators. This is reminiscent of the Nielsen-Chadha counting [2,3]. In fact, we still find that, for even dispersion relations, the number of broken generators is twice the number of Goldstones, just in the broader sense that gapped states have an “even” dispersion relation, i.e., $E(p) \sim |\vec{p}|^0 + \mathcal{O}(p^2)$. An intuitive picture of what is going on is provided by the following example.

The linear SO(3) sigma model.—Consider a Lagrangian with internal symmetry SO(3), linearly realized on a scalar triplet $\vec{\phi}$:

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \frac{1}{2} m^2 |\vec{\phi}|^2 - \frac{1}{4} \lambda |\vec{\phi}|^4. \quad (16)$$

Let us pick one of the generators of SO(3), say, $\tau = \tau_3$, where τ_i generates rotations about the ϕ_i axis, and let us consider the theory at finite charge density for the corresponding charge Q . The standard way to build the nonrelativistic Lagrangian at finite density—see, e.g., Refs. [10,14]—is to introduce a constant nondynamical gauge field pointing in the time direction, which in our case amounts to the replacement

$$-\partial_\mu \Phi^\dagger \partial^\mu \Phi \rightarrow (\partial_0 - i\mu) \Phi^\dagger (\partial_0 + i\mu) \Phi - \partial_j \Phi^\dagger \partial_j \Phi, \quad (17)$$

where Φ is the complex combination $\Phi = \phi_1 + i\phi_2$. After going to polar coordinates, $\Phi = \sigma e^{i\theta}$, we find

$$\begin{aligned} \mathcal{L}_{(\mu)} = & -\frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} \sigma^2 \partial_\mu \theta \partial^\mu \theta - \frac{1}{2} \partial_\mu \phi_3 \partial^\mu \phi_3 \\ & - \frac{1}{2} m^2 (\sigma^2 + \phi_3^2) - \frac{1}{4} \lambda (\sigma^2 + \phi_3^2)^2 \\ & + \mu \sigma^2 \dot{\theta} + \frac{1}{2} \mu^2 \sigma^2. \end{aligned} \quad (18)$$

The ground state at finite chemical potential corresponds to constant field solutions of this Lagrangian. While we always have $\langle \phi_3 \rangle = 0$, there are cases when $\langle \sigma \rangle \neq 0$, which spontaneously breaks the symmetry generated by Q . This happens when the configuration $\vec{\phi} = 0$ was unstable to begin with ($m^2 < 0$) or when the chemical potential exceeds a critical value ($\mu^2 > m^2$).

In the broken phase, the vacuum expectation value of the radial field is $\langle \sigma \rangle^2 = \frac{1}{\lambda} (\mu^2 - m^2)$. Expanded to second order about this configuration, the Lagrangian (18) reads

$$\begin{aligned} \mathcal{L}_{(\mu)} \simeq & -\frac{1}{2} \partial_\mu \delta \sigma \partial^\mu \delta \sigma - \frac{1}{2} \langle \sigma \rangle^2 \partial_\mu \theta \partial^\mu \theta \\ & - \frac{1}{2} \partial_\mu \phi_3 \partial^\mu \phi_3 + 2\mu \langle \sigma \rangle \dot{\theta} \delta \sigma \\ & - (\mu^2 - m^2) \delta \sigma^2 - \frac{1}{2} \mu^2 \phi_3^2. \end{aligned} \quad (19)$$

Let us assume that $\langle \vec{\phi} \rangle$ points in the ϕ_1 direction. Of the whole SO(3) symmetry group, the only residual symmetry is that associated with τ_1 —rotations about ϕ_1 . This symmetry breaking pattern would normally be associated with two massless Goldstone bosons, one for each broken generator. Instead, here we see that the would-be Goldstone field ϕ_3 associated with τ_2 has acquired a mass $m_3 = \mu$, in agreement with our general result, since τ_2 does not commute with τ_3 . The angular field (θ) is massless and can be identified with the Goldstone boson associated with τ_3 , which obviously commutes with itself.

In the alternative (but equivalent) SSP language [11], one starts from the relativistic Lagrangian (16) and looks for a *time-dependent* background solution that rotates about the ϕ_3 axis, with $\dot{\vec{\phi}} = \mu \tau_3 \cdot \vec{\phi}$. This just corresponds to a constant speed in the angular field, $\theta(x) = \mu t$. Such solutions are allowed only for $m^2 < 0$ or for $\mu^2 > m^2$, in agreement with what we found above. After expanding in θ fluctuations about such a solution, $\theta \rightarrow \mu t + \theta$, one finds precisely the Lagrangian (18) and its quadratic approximation (19), with the same excitation spectrum as above. However, the SSP language is closer to the viewpoint we emphasized in this Letter because it makes manifest that having a finite charge density for a spontaneously broken charge necessarily implies a spontaneous breakdown of time translations as well. Moreover, it stresses that Lorentz invariance is broken spontaneously by the field configuration one considers, rather than at the level of the Lagrangian. And finally, it gets the breaking pattern for internal symmetries right: *All* SO(3) generators are broken,

albeit in a time-dependent fashion, in the sense that $\langle[\tau_1, \vec{\phi}(x)]\rangle$ and $\langle[\tau_2, \vec{\phi}(x)]\rangle$ depend on time. This is what one would discover from our general analysis above, if one evaluated the order parameters A_I at generic positions x rather than at the origin.

Concluding remarks.— We conclude by stressing that our derivation of Eq. (15) involved no approximation. As a result, our expression for the gap is an exact nonperturbative prediction. Our results evade the Nielsen-Chadha theorem because one of its (implicit) assumptions is violated, namely, that Q commutes with all other broken charges. This crucial difference should be taken into consideration also when comparing our results with the literature, like for instance the “kaon condensation” model of Refs. [4,5]. On the other hand, our results agree with the theorem of Ref. [5], which states—among other things—that there are no subtleties in counting the gapless Goldstones for relativistic theories as long as all charge densities vanish.

The work of A. N. is supported by the DOE (DE-FG02-92-ER40699) and by NASA ATP (09-ATP09-0049).

-
- [1] R. V. Lange, *Phys. Rev. Lett.* **14**, 3 (1965).
 - [2] H. B. Nielsen and S. Chadha, *Nucl. Phys.* **B105**, 445 (1976).
 - [3] T. Brauner, *Symmetry* **2**, 609 (2010).

- [4] V. A. Miransky and I. A. Shovkovy, *Phys. Rev. Lett.* **88**, 111601 (2002).
- [5] T. Schafer, D. T. Son, M. A. Stephanov, D. Toublan and J. J. M. Verbaarschot, *Phys. Lett. B* **522**, 67 (2001).
- [6] H. Watanabe and T. Brauner, [arXiv:1109.6327](https://arxiv.org/abs/1109.6327).
- [7] J. Goldstone, A. Salam, and S. Weinberg, *Phys. Rev.* **127**, 965 (1962).
- [8] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, England, 1996), Vol. 2.
- [9] In general, the right-hand side should read $\lambda|\mu\rangle$, with nonvanishing λ . From applying the familiar thermodynamic relation $E + PV = TS + \mu Q$ to our zero-temperature system, we see that λ is related to the pressure. Such a charge density breaks Lorentz invariance because it transforms like the time component of a four-vector. In the following, we will drop λ from our formulas to make them less cluttered, although keeping track of it is straightforward. Formally, for any given μ , we can set λ to zero by adjusting the cosmological constant.
- [10] M. Le Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).
- [11] A. Nicolis and F. Piazza, *J. High Energy Phys.* **06** (2012) 025.
- [12] This is the removal of one of the Nielsen-Chadha assumptions to which we alluded.
- [13] Once the algebra for the charges is given, the charge-current commutators are uniquely determined up to possible contact terms that vanish at zero momentum, i.e., total spacial derivatives. Since we took the $\vec{p} \rightarrow 0$ limit, such possible extra terms will not matter for us.
- [14] D. T. Son, [arXiv:hep-ph/0204199](https://arxiv.org/abs/hep-ph/0204199).