Leitner, Phys. Rev. 112, 273 (1958).

[7]Except when specified otherwise, components of vectors and tensors refer to any right-handed set of Cartesian coordinate axes chosen without reference to the Y^* decay.

[8]Details of the derivation are available in a University of California, Los Angeles, California, preprint (unpublished). Using helicity states for the decay products, these results are obtained using standard methods [see, e.g., reference 3 and M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) 7, 404 (1959)]. We thank S. Berman for pointing this out.

[9]If the Y^* state is formed from all Y^* produced with momentum $\vec{u}'$ and $\vec{u}$ is the incident momentum, the $t_L{}^M$ are polynomials in the components of $\hat{u}$ and $\hat{u}'$: see, e.g., H. H. Joos, Forstchr. Physik 10, 65 (1962).

[10]Using (20) and (21), one may evaluate the $t_L{}^M$ when ρ is known; for an example, see R. K. Adair, Phys. Rev. 100, 1540 (1955). In the Adair analysis, only the $t_L{}^M$ with $M=0$ and L even are different from zero. If parity is violated in the decay, a unique function $I\vec{P}\cdot\vec{k}$ is obtained for each J [see Eq. (12′)].

UPPER LIMITS FOR COUPLING CONSTANTS IN QUANTUM FIELD THEORY

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In quantum field theory it is usually considered that the magnitudes of the coupling constants can assume any value irrespective of the particle mass. In the case of a bound state in the nonrelativistic theory, however, Heisenberg[1] showed that the coupling constant is expressed through the constant in the asymptotics of the wave function of this state. Subsequently in a number of reports[2-8] it was shown that the physical (renormalized) coupling constants are bounded above by definite boundaries which depend on the particle mass. In these reports, however, the considerations either had resort to models or[5] were based on the assumption that the interaction of the elementary particles is characterized by an effective radius, which does not increase with the increase of the coupling constant. We will show here that the restrictions on the magnitudes of the coupling constants at any given mass follow from the general principles of quantum field theory with no additional assumption. (A detailed report appears elsewhere.[9])

Our assumption will be based on the representation of Green's function in the form of the Lehman-Källén[10] expansion; i.e., we will consider that all the conditions under which this expansion takes place are fulfilled.

We will first consider a case of interaction of three boson fields with zero spin a, b, and c, and we will obtain the restriction on the magnitude of the coupling constant g^2 of these three fields. We will consider that particles a, b, and c are stable, $m_a < m_b < m_c$; we will assume also that b and c are the closest particles (with regard to total mass) to particle a. Based on the Lehman-Källén expansion of the Green's function of boson a in our previous work,[7] we obtain the following inequality restricting the possible magnitude of the renormalized coupling constant g^2:

$$\frac{1}{4\pi}\frac{g^2}{(m_b+m_c)^2}\Phi < 1,$$

$$\Phi = 2(m_b+m_c)^2\int_{(m_b+m_c)^2}^{\infty}\frac{|\Gamma(\kappa^2)|^2}{(\kappa^2-m_a{}^2)^2}\frac{q(\kappa^2)}{\kappa}d\kappa^2. \quad (1)$$

Here $\Gamma(\kappa^2) \equiv \Gamma(\kappa^2, m_b{}^2, m_c{}^2)$ is the vertex part for a, b, c particle interaction; $q(\kappa^2) = (2\kappa)^{-1}[\kappa^2-(m_b+m_c)^2]^{1/2}[\kappa^2-(m_b-m_c)^2]^{1/2}$ is the momentum of b and c particles in the center of mass. In order to obtain from inequality (1) definite restriction for g^2, the magnitude of Φ at given masses m_a, m_b, m_c must be bounded below.

Let us find the minimum in the class of functions $\Gamma(\kappa^2)$ having the following characteristics:

(I) $\Gamma(\kappa^2)$ is a holomorphic function of κ^2 in the complex κ^2 plane with a cut along the real axis, extending from point $\kappa^2=(m_b+m_c)^2$ to infinity. To the left of the point $\kappa^2=(m_b+m_c)^2$ on the real axis, $\Gamma(\kappa^2)$ is real.

(II) The rate of increase of $\Gamma(\kappa^2)$ as κ^2 tends to ∞ is not more rapid than an exponential increase.

(III) At the point $\kappa^2=m_a{}^2$, $\Gamma(m_a{}^2)=1$.

We assume that $\Gamma(\kappa^2)$ has no poles in the complex plane. In principle, the poles of $\Gamma(\kappa^2)$ could be situated on the real axis in the interval $m_a{}^2 < \kappa^2 < (m_b+m_c)^2$ at point κ^2 at which the Green's function $D(\kappa^2)$ becomes zero, and $\Gamma(\kappa^2)D(\kappa^2) \to$ const when $\kappa^2 \to \kappa_n{}^2$. [The latter case follows from, for example, Schwinger's equation for the

Green's function $D(\kappa^2)$.] These poles of $\Gamma(\kappa^2)$ correspond to bound states of the particles b and c, and because of the term $g^2\Gamma^2(\kappa^2)D(\kappa^2)$, will lead to poles in the scattering amplitudes of particles b and c. [This takes place, for example, in the theory of superconductivity where the pole of $\Gamma(\kappa^2)$ corresponds to the bound state of Cooper's pair.] Thus the result of our assumption is that there are no particles with masses intermediate between m_a and m_b+m_c.

The characteristics (I)-(III) will be fulfilled if we write the dispersion relation for $\Gamma(\kappa^2)$:

$$\Gamma(\kappa^2) = 1+\frac{\kappa^2-m_a^2}{\pi}\int_{(m_b+m_c)^2}^{\infty} d\kappa'^2 \frac{\operatorname{Im}\Gamma(\kappa'^2)}{(\kappa'^2-\kappa^2-i\delta)(\kappa'^2-m_a^2)}, \tag{2}$$

with an arbitrary $\operatorname{Im}\Gamma(\kappa^2)$. [It can be shown[9] that one subtraction in (2) is sufficient.] Then substituting (2) in (1), one can find the minimum of the functional $\Phi[\Gamma(\kappa^2)]$ in the class of functions $\Gamma(\kappa^2)$ which is of interest to us, by varying Φ over $\operatorname{Im}\Gamma(\kappa^2)$. For $\operatorname{Im}\Gamma(\kappa^2)$ we obtain Fredholm's integral equation of the second kind, which has only one solution. Knowing $\operatorname{Im}\Gamma(\kappa^2)$, one can find the function $\Gamma(\kappa^2)$ minimizing Φ and the value of $\Phi_{\min}$. Unfortunately, the resulting integral equation is rather complex, and in a generalized case it can be solved only numerically on an electronic computer (see reference 9).

The minimum value of Φ can be also obtained by an analytical method.[11-13] Let us introduce the notations

$$x=\kappa^2/(m_b+m_c)^2,$$

$$\alpha=m_a^2/(m_b+m_c)^2,$$

$$\lambda=(m_b-m_c)^2/(m_b+m_c)^2.$$

In order to find the minimum of the integral

$$\Phi=\int_1^{\infty}\frac{(x-1)^{1/2}(x-\lambda)^{1/2}}{x(x-\alpha)^2}|\Gamma(x)|^2dx, \tag{3}$$

in a complex x plane, conformal transformation is carried out:

$$z=-[(x-1)^{1/2}-i(1-\alpha)^{1/2}]/[(x-1)^{1/2}+i(1-\alpha)^{1/2}], \tag{4}$$

converting the two sides of the cut along the real axis from 1 to infinity into a unit circle. All the complex x plane with the cut passes into the inner part of the unit circle and the point $x-\alpha$ passes into the center of the circle. The integral (3) is transformed into

$$\Phi=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)|\Gamma(z)|^2d\theta, \quad z=e^{i\theta},$$

$$f(\theta)=\pi(1-\alpha)^{-1/2}u[1-\lambda+(1-\alpha)u]^{1/2}/[1+(1-\alpha)u](1+u),$$
$$u=\tan^2(\tfrac{1}{2}\theta). \tag{5}$$

$\Gamma(z)$ is an analytic function inside the unit circle and $\Gamma(0)=1$. The solution of the problem of finding the minimum of integral (5) over the class of functions $\Gamma(z)$ which are analytic in a circle when $\Gamma(0)=1$ is carried out by the expansion of $\Gamma(z)$ over the system of polynomials which are orthogonal on a unit circle with the weight $f(\theta)$. The answer is as follows:

$$\Phi_{\min}=\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln f(\theta)d\theta\right]. \tag{6}$$

After elementary integration, we obtain

$$\Phi_{\min}=\tfrac{1}{4}\pi[(1-\lambda)^{1/2}+(1-\alpha)^{1/2}]/(1-\alpha)^{1/2}[1+(1-\alpha)^{1/2}]^2. \tag{7}$$

Thus in the case of interaction of three boson fields, the coupling constant g^2 is restricted by the inequality

$$g^2<16\frac{[(m_b+m_c)^2-m_a^2]^{1/2}\{m_b+m_c+[(m_b+m_c)^2-m_a^2]^{1/2}\}^2}{2(m_bm_c)^{1/2}+[(m_b+m_c)^2-m_a^2]^{1/2}}. \tag{8}$$

In the nonrelativistic case when the binding energy $\Delta=m_b+m_c-m_a\ll m_a$, (8) goes over into the expression obtained previously[7] (μ is the reduced mass of particles b and c):

$$g^2<16\,m_a^2(\Delta/2\mu)^{1/2}. \tag{9}$$

The fact that at low binding energies g^2 is proportional to $\sqrt{\Delta}$ has been noted in a number of reports[1-4,6] in which the considerations were based on the nonrelativistic theory, and it was assumed that the range of nuclear forces equaled zero.

With these assumptions the authors of references 1-4 and 6 arrived at limit on g^2 two times smaller than ours (this restriction is obtained if we assume that $\Gamma=1$). In reference 4 it was noted that the calculation of the finiteness of the range of nuclear forces diminishes the restriction on the coupling constant g^2. From our discussion it follows that because of the finiteness of the range of nuclear forces, the magnitude of g^2 can increase not more than two times. In the case of a deuteron (a = deuteron, b and c = neutron and proton), from experimental data on neutron-proton scattering $g^2 = 12 m_D^2(\Delta/2\mu)^{1/2}$; i.e., for a deuteron the estimate (9) is very close to the actual value.

Let us now consider a case when a and b are fermions with spin $\frac{1}{2}$, and c is a boson with spin zero. Let us write the Lehman-Källén representation of the Green's function for fermion a:

$$G(p) = \frac{1}{\not{p} - m_a} - \int_{(m_b+m_c)^2}^{\infty} \frac{(p+\kappa)\rho_1(\kappa^2) - \rho_2(\kappa^2)}{\kappa^2 - p^2 - i\delta} d\kappa^2,$$

$$2\kappa\rho_1 \geqslant \rho_2 \geqslant 0, \tag{10}$$

and we will assume that $G(\not{p}) = \not{p} f_1(p^2) + f_2(p^2)$. Then f_1^{-1} will be an R function, and proceeding in the same manner as in reference 7, we will obtain

$$\int_{(m_b+m_c)^2}^{\infty} \frac{\rho_1(\kappa^2)}{|f_1(\kappa^2)|^2} \frac{d\kappa^2}{(\kappa^2 - m_a^2)^2} < 1. \tag{11}$$

Increasing the inequality, we will calculate in ρ_1 only the contribution of the two-particle states $b+c$. Then ρ_1 will be expressed by the vertex part which will have the form

$$\Gamma(\not{p}) = \Gamma_1(p^2) + \not{p}\Gamma_2(p^2), \tag{12}$$

if boson c is scalar and

$$\Gamma(\not{p}) = \gamma_5\Gamma_1(p^2) + \not{p}\gamma_5\Gamma_2(p^2) \tag{12'}$$

if boson c is pseudoscalar. After substituting the corresponding expression for $\rho_1(\kappa^2)$, the inequality (11) will have the form $(g^2/2\pi)\Phi < 1$:

$$\Phi = \int_{(m_a+m_b)^2}^{\infty} \frac{d\kappa^2}{(\kappa^2 - m_a^2)^2\kappa^2} \frac{q(\kappa^2)}{\kappa}$$
$$\times\{|F_1(\kappa^2)\kappa + F_2(\kappa^2)m_a|^2[(\kappa+m_b)^2 - m_c^2]$$
$$+ |F_1(\kappa^2)\kappa - F_2(\kappa^2)m_a|^2[(\kappa \mp m_b)^2 - m_c^2]\}, \tag{13}$$

$$F_1 = (f_1\Gamma_1 + f_2\Gamma_2)/2f_1, \quad F_2 = (f_1\Gamma_2 p^2 + f_2\Gamma_1)/2f_1 m_a \tag{14}$$

(the upper sign refers to the scalar case, the lower sign to pseudoscalar case). In accordance with the definition of physical charge, the following condition will exist:

$$\Gamma_1(m_a^2) + m_a\Gamma_2(m_a^2) = 1. \tag{15}$$

We will assume that the functions $\Gamma_1(\kappa^2)$ and $\Gamma_2(\kappa^2)$ have the same analytical characteristics (I) and (II) as the function $\Gamma(\kappa^2)$ in the boson case. Specifically, $\Gamma_1(\kappa^2)$ and $\Gamma_2(\kappa^2)$ do not have poles in the complex κ^2 plane and, consequently, the Green's function of fermion a has no zeros, connected with the poles of Γ_1 and Γ_2. Also we will assume here that the Green's function $f_1(p^2)$, in general, has no zeros. (It can be shown[14] that the rejection of this assumption will not change the deduction regarding the restriction of the constant g^2.) Then the minimum of the functional Φ exists and can be found by the same method as in the boson case. In finding the minimum it is convenient to assume that

$$F_1(\kappa^2) = \frac{2m_b(m_b+m_c)}{\kappa^2 + m_b^2 - m_c^2}[\mp\bar{F}_1(\kappa^2) + \bar{F}_2(\kappa^2)],$$

$$F_2(\kappa^2) = \frac{m_b+m_c}{m_a}\bar{F}_1(\kappa^2).$$

It follows from (15) that at point $\kappa^2 = m_a^2$ the functions $\bar{F}_1(\kappa^2)$ and $\bar{F}_2(\kappa^2)$ satisfy the conditions

$$\bar{F}_1(m_a^2) = \frac{m_a}{2(m_b+m_c)};$$

$$\bar{F}_1(m_a^2) = \frac{[(m_a^2 + m_b^2 - m_c^2)/2m_b] \pm m_a}{2(m_b+m_c)}. \tag{16}$$

From the regularity requirements on $F_1(\kappa^2)$ at the point $\kappa^2 = -m_b^2 + m_c^2$, there follows another condition for the functions $\bar{F}_1$ and $\bar{F}_2$:

$$\mp\bar{F}_1(-m_b^2 + m_c^2) + \bar{F}_2(-m_b^2 + m_c^2) = 0. \tag{16'}$$

Just as in the case for bosons, the minimum of the functional Φ can be found by writing down the dispersion relations for $\bar{F}_1$ and $\bar{F}_2$, substituting them in (14), and minimizing over $\mathrm{Im}\bar{F}_1$ and $\mathrm{Im}\bar{F}_2$ in conditions (16) and (16'). The integral equations obtained can be solved numerically. In order to obtain an analytic expression for $\Phi_{\min}$, let us carry out the conformal transformation (4).

Then

$$\Phi=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\{f_1(\theta)|\bar{F}_1(z)|^2+f_2(\theta)|\bar{F}_2(z)|^2\},\quad z=e^{i\theta};$$

$$f_1(\theta)=\pi(1-\alpha)^{1/2}\frac{u^2}{1+u}\times\frac{[1-\lambda+(1-\alpha)u]^{3/2}}{[1+(1-\alpha)u]^2[1+\lambda^{1/2}+(1-\alpha)u]},\qquad(17)$$

$$f_2(\theta)=\frac{\pi(1+\lambda^{1/2})^2}{(1-\alpha)^{1/2}}\frac{u}{1+u}\times\frac{[1-\lambda+(1-\alpha)u]^{1/2}}{[1+(1-\alpha)u][1+\lambda^{1/2}+(1-\alpha)u]};\quad u=\tan^2(\tfrac{1}{2}\theta).$$

Let us expand the functions $\bar{F}_1(z)$ and $\bar{F}_2(z)$ in the unit circle $|z|<1$ over the polynomial systems $\psi_n^{(1)}$ and $\psi_n^{(2)}(z)$ orthogonal in a unit circle with weights $f_1(\theta)$ and $f_2(\theta)$, respectively: $\bar{F}_1(z)=\sum a_n\psi_n^{(1)}(z)$; $\bar{F}_2(z)=\sum b_n\psi_n^{(2)}(z)$. Substituting these expansions in (17) and finding the minimum in (17) under the conditions (16) and (16′), after certain calculations with the aid of formulas for the sums of orthogonal polynomials from reference 12 we find

$$\Phi_{\min}=\frac{\pi}{16}\frac{(v+w)^3}{(1+v)^2(v+y)}\left\{\frac{\alpha v}{(1+v)^2}+\frac{(1\pm\sqrt{\alpha})^2(\sqrt{\lambda}\pm\sqrt{\alpha})^2}{v(v+w)^2}+2\frac{(w+y)^2(v+y)^2}{(\alpha+\sqrt{\lambda})^2(w+2y+y^2)}\times\left[\frac{\alpha+\sqrt{\lambda}}{v+w}\pm\alpha^{1/2}\left(\frac{1+\sqrt{\lambda}}{v+w}-\frac{vy(1+y)}{(v+y)(1+v)}\right)\right]^2\right\},\qquad(18)$$

where $v=(1-d)^{1/2}$, $w=(1-\lambda)^{1/2}$, and $y=(1+\lambda^{1/2})^{1/2}$. In the most interesting case, which is the interaction of pions with nucleons, by substituting in (18) the numerical mass values we have $\Phi_{\min}=0.0245$ and $g_{NN\pi}^2<85$. For the constant for $\Sigma\Lambda\pi$ interaction for opposite parities of Σ and Λ, we obtain $g_{\Sigma\Lambda\pi}^2<3.2$ (a numerical solution of the integral equations gives, naturally, the same results).

The fact that the magnitude of the coupling constant g^2 cannot be greater than a certain value $g_{\max}^2$ signifies that the amplitudes of quantum field theory, considered as functions of g^2, have a singularity point when $g^2=g_{\max}^2$, and when $g^2>g_{\max}^2$ theory will be physically contradictory. Therefore all attempts at solving the problems by using an expansion in powers of $1/g^2$ at given masses (strong coupling theory) have very little chance for success.

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