# SYMMETRY FROM BOOTSTRAP MECHANISM*

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Recently, several authors,[1-4] notably Cutkosky[1] and Capps,[2] have explored the suggestion that approximate higher symmetry in strong interaction need not be assumed but may be a consequence of dynamics. The relation between these works, however, has not been elucidated. In this note, we derive from the $N/D$ bootstrap mechanism equations relating the coupling constants which are generalizations of certain equations considered by Cutkosky for a special case of mutually interacting vector mesons. We shall then apply these equations to some special cases, including the case considered by Capps.[2] We conclude that the $G_2$ symmetry[5] is not consistent with the bootstrap mechanism here employed.

Consider the interaction of $N$ scalar or pseudoscalar mesons all of mass $m$ through the exchange of $M$ vector mesons all of mass $\mu$. The most general coupling is

$$\sum_{abr} g_{ab}{}^{r}(p_a - p_b)\phi_a\phi_b A_r, \tag{1}$$

where without loss of generality, all fields are taken real and the coupling constants $g_{ab}{}^{r}$ antisymmetric in the indices $a, b$. We write the $p$-wave amplitude $T_{ab,cd}$ for the process $ab \to cd$ in the usual $ND^{-1}$ form,

$$T_{ab,cd} = \sum_{ef} N_{ab,ef} D_{ef,cd}{}^{-1}. \tag{2}$$

Note that the sum is over the different channels, i.e., over couples $(ef)$ only.

Following Zachariasen and Zemach,[6] we approximate $N(x)$ by the Born approximations, namely,

$$N_{ab,cd} = \sum_r (g_{ac}{}^{r} g_{bd}{}^{r} - g_{ad}{}^{r} g_{bc}{}^{r}) F(x) = V_{ab,cd} F(x), \tag{3}$$

where $F(x)$ is some function of the energy the exact form of which we shall not need. This gives, in matrix notation,

$$D(x) = 1 - \alpha(x) V,$$
$$N(x) = F(x) V,$$
$$T(x) = F(x) V[I - \alpha(x) V]^{-1}; \tag{4}$$

where

$$\alpha(x) = \frac{x - x_t}{\pi}\int_1^\infty \frac{F(x')dx'}{(x'-x)(x'-x_t)}.$$

The bootstrap mechanism requires that the $M$ vector mesons which are exchanged to give the "potential" $V$ should occur also as poles in $T$ in the direct channel. Hence $T$ should have an $M$-fold pole at $x = \mu$ with residue proportional to $\sum_r g_{ab}{}^{r} g_{cd}{}^{r}$. Now $V$, being real and symmetric, has real eigenvalues. Let $\psi^1, \psi^2, \cdots$ be a complete orthonormal set of its eigenvectors, so that the real orthogonal matrix

$$O = (\psi^1, \psi^2, \cdots) \tag{5}$$

diagonalizes $V$:

$$O^{-1} V O(\lambda_1, \lambda_2, \cdots) = \Lambda. \tag{6}$$

By (4), $O$ also diagonalizes $N$, $D$, and $T$. Hence

$$T = F(x) O \Lambda [I - \alpha(x)\Lambda]^{-1} O^{-1}. \tag{7}$$

The condition that $T$ has an $M$-fold pole at $x = \mu$ then requires that there be an exactly $M$-fold degenerate eigenvalue $\lambda = 1/\alpha(\mu)$. It can readily be seen from (7) that the residue of $T$ at $x = \mu$ is given by

$$\mathrm{Res}\, T_{ab,cd} = C(\mu) \sum_{r=1} \psi_{ab}{}^{r} \psi_{cd}{}^{r}, \tag{8}$$

where $C(\mu)$ is a constant independent of the indices and $\psi^1, \cdots, \psi^M$ are the eigenvectors belonging to value $\lambda$. The bootstrap condition therefore requires

$$\sum_{r=1}^{M} \psi_{ab}{}^{r} \psi_{cd}{}^{r} \sim \sum_{r=1}^{M} g_{ab}{}^{r} g_{cd}{}^{r}. \tag{9}$$

This implies that the coupling constants $g_{ab}{}^{r}$ themselves may be taken as components of orthonormal eigenvectors of the potential matrix, apart from a common normalization factor which we may choose as unity. Using the antisymmetry of the $g$'s, we may then write

$$\sum_{r,c,d}(g_{ac}{}^{r}g_{bd}{}^{r}-g_{ad}{}^{r}g_{bc}{}^{r})g_{cd}{}^{s}=\lambda g_{ab}{}^{s}, \qquad (10)$$

$$\sum_{ab} g_{ab}{}^{r}g_{ab}{}^{s}=\delta_{rs}; \qquad (11)$$

or equivalently, instead of (10),

$$2\sum_{rdc} g_{ac}{}^{r}g_{bd}{}^{r}g_{cd}{}^{s}=\lambda g_{ab}{}^{s}, \qquad (10')$$

where the sums are now free sums over all indices. Equations (10′) and (11) are generalizations of the equations considered by Cutkosky[1] for his special model. We note that the same Eqs. (10′) and (11) can be obtained if the vector mesons were replaced by scalar mesons, except that the $g$'s are then symmetric instead of antisymmetric in the two lower indices.

Equations (10′) and (11) can be represented diagrammatically as in Fig. 1. (It is then readily seen that these equations imply Sakurai's[3] condition for degeneracy of the vector-meson self-energy up to the fourth order.)

We shall consider three special cases as examples:

(I) Mutual interaction of $\pi^+, \pi^0, \pi^-$ bootstrapping $\rho^+, \rho^0, \rho^-$. Form the usual real fields $\pi_i$ and $\rho_i$, $i=1,2,3$. Charge-conjugation invariance and charge conservation together with Eq. (11) imply

$$(g_{23}{}^{1})^2=(g_{31}{}^{2})^2=(g_{12}{}^{3})^2 \qquad (12)$$

with all other coupling constants zero, as is obtained from isospin symmetry. Equation (10′) is satisfied identically.

(II) Mutual interaction of eight pseudoscalar mesons $(\pi, K, \eta)$ bootstrapping eight vector mesons $(\rho, K^*, \omega)$. As in reference 2, we neglect mass differences within multiplets and assume isospin and charge-conjugation invariance as well as conservation of charge and strangeness. Equation (11) gives

$$(\gamma_{\pi\pi}{}^{\rho})^2+(\gamma_{KK}{}^{\rho})^2=1,$$

$$(\gamma_{KK}{}^{\omega})^2=1,$$

$$(\gamma_{K\pi}{}^{K*})^2+(\gamma_{K\eta}{}^{K*})^2=1; \qquad (13)$$

while Eq. (10′) gives

$$\tfrac{3}{2}(\gamma_{KK}{}^{\rho})^2+\tfrac{1}{2}(\gamma_{KK}{}^{\omega})^2=\lambda,$$

$$(\gamma_{\pi\pi}{}^{\rho})^2+\frac{2\sqrt{2}}{3}\frac{\gamma_{KK}{}^{\rho}(\gamma_{K\pi}{}^{K*})^2}{\gamma_{\pi\pi}{}^{\rho}}=\lambda,$$

$$(\gamma_{K\eta}{}^{K*})^2+(\gamma_{\pi\pi}{}^{K*})^2=\lambda,$$

$$-\tfrac{1}{2}(\gamma_{KK}{}^{\rho})^2+\tfrac{1}{2}(\gamma_{KK}{}^{\omega})^2+\frac{2\sqrt{2}}{3}\frac{\gamma_{\pi\pi}{}^{\rho}(\gamma_{K\pi}{}^{K*})^2}{\gamma_{KK}{}^{\rho}}=\lambda,$$

$$\sqrt{2}\gamma_{KK}{}^{\rho}\gamma_{\pi\pi}{}^{\rho}-\tfrac{1}{3}(\gamma_{K\pi}{}^{K*})^2+(\gamma_{\eta K}{}^{K*})^2=\lambda; \qquad (14)$$

all other conditions being automatically satisfied. The two sets of Eqs. (13) and (14) are readily shown to be consistent and yield

$$(\gamma_{\pi\pi}{}^{\rho})^2:(\gamma_{KK}{}^{\rho})^2:(\gamma_{KK}{}^{\omega})^2:(\gamma_{\pi K}{}^{K*})^2:(\gamma_{\eta K}{}^{K*})^2=\tfrac{4}{3}:\tfrac{2}{3}:2:1:1,$$

$$\gamma_{\pi\pi}{}^{\rho}\gamma_{KK}{}^{\rho}>0;$$

identical with the result of reference 2 and consistent with $SU_3$ symmetry.

(III) Mutual interaction of seven pseudoscalar mesons $(\pi, K)$ bootstrapping seven vector mesons $(\rho, K^*)$, i.e., omitting $\eta$ and $\omega$ in (II). This case has some interest since it is connected with the seven-dimensional representation of $G_2$. There

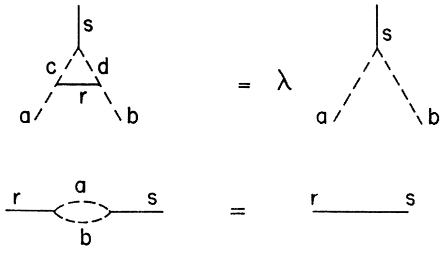


FIG. 1. Diagrams representing Eqs. (10′) and (11).

is again a unique solution:

$$(\gamma_{\pi\pi}{}^{\rho})^2 : (\gamma_{KK}{}^{\rho})^2 : (\gamma_{\pi K}{}^{K*})^2 = \tfrac{1}{3} : \tfrac{2}{3} : 1,$$

$$\gamma_{\pi\pi}{}^{\rho} \gamma_{KK}{}^{\rho} < 0;$$

but now $\lambda = -1$ whereas in cases (I) and (II), $\lambda = +1$. It can be shown that a negative $\lambda$ corresponds to a repulsive force and the equation $\lambda = 1/\alpha(\mu)$ is not satisfied. This means that this case is not consistent with the $N/D$ bootstrap approximation employed here.

It is difficult at this stage to estimate the exact physical significance of the several special cases quoted above. The authors believe that the equations derived are quite general and may well be arrived at by various arguments. It is interesting, however, that they can be derived from $N/D$ bootstrap which is at present the only available tool with a reasonable degree of success in strong interactions. The general group-theoretical properties of these equations are also of interest and are under investigation.

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[3]J. J. Sakurai, Phys. Rev. Letters 10, 446 (1963).

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[6]F. Zachariasen and C. Zemach, Phys. Rev. 128, 849 (1962).

## ERRATUM

CYCLOTRON RESONANCE INVOLVING CURRENT SHEETS IN ALUMINUM. C. C. Grimes, A. F. Kip, F. Spong, R. A. Stradling, and P. Pincus [Phys. Rev. Letters 11, 455 (1963)].

The scale on the ordinate for the upper curve of Fig. 2 should be multiplied by four.