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[1]See J. Bernstein, M. Ruderman, and G. Feinberg, Phys. Rev. 132, 1227 (1963), for a discussion of the present experimental limits on the charge, mass, etc., of the neutrinos.

[2]T. D. Lee, Phys. Rev. 128, 899 (1962). Throughout the present paper, all unexplained notations are the same as those in this reference.

[3]T. D. Lee and C. N. Yang, Phys. Rev. 128, 885 (1962).

[4]This is to be contrasted with a similar relation for a charged particle, which is valid only if both the initial and the final wave functions satisfy the correct Dirac equation.

[5]This can be seen by considering a three-point W-W-γ vertex $V_\lambda(p,p')$ where both the initial and final W lines are virtual. If $\kappa \neq 0$, then both of these W propagators can assume their singular form $[p\tilde{p}/(p^2+m^2)]$ at large p. However, $\tilde{p}'V_\lambda(p',p)p=0$ if $\kappa=0$. Therefore, for $\kappa=0$ only one of these two propagators can take on the singular form. The relevant expansion parameter for $\kappa=0$ is $(\alpha\Lambda^2)$, and that for $\kappa\neq 0$ is $(\alpha\kappa^2\Lambda^4)$.

[6]The same considerations have been used in reference 2 to calculate the radiative corrections to the electromagnetic properties of $W^\pm$ for $\kappa\neq 0$. Similar calculations for $\kappa=0$ will be given in a separate publication.

[7]For a general discussion of degenerate systems and mass singularities, see T. D. Lee and M. Nauenberg, Phys. Rev. (to be published).

[8]See G. Danby et al., Phys. Rev. Letters 10, 260 (1963).

[9]See G. Feinberg and A. Pais, Phys. Rev. 131, 2724 (1963).

RESONANCE MULTIPLETS AND BROKEN SYMMETRY*

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In this Letter we describe some experimental and theoretical consequences of the fact that resonance poles may appear on more than one Riemann sheet of the S matrix. We have in mind particularly those resonances which belong to multiplets in the unitary symmetry scheme SU_3. The main points we wish to note are these:

(i) In the S matrix for two or more coupled channels a resonance may appear as a pole on more than one of the unphysical Riemann sheets.

(ii) In general, only one of these poles (the dominant pole) will be near to the physical region. Under certain circumstances, however, two poles may be comparably important, in which case interference between the poles could have an observable effect on the position and shape of the resonance.

(iii) In the case of resonance multiplets of the approximate symmetry scheme SU_3, the presence of several poles on different sheets representing a single resonance allows the members of each multiplet to move into coincidence, when full symmetry is established, without any of the difficulties discussed by Oakes and Yang.[1]

The circumstances under which a resonance or bound state leads to poles on more than one Riemann sheet can be examined in terms of analyticity and unitarity of the S matrix.[2] This examination will not be given here, but we wish to note that in addition to the usual assumptions of S-matrix theory, our work requires analyticity in the coupling between different channels.[3] For the sake of brevity our discussion is given, instead, in terms of a simple resonance model based on a sum of self-energy diagrams.[4] We should, however, emphasize that our results are more general than the particular model considered. In particular, the fact that the model is S wave is not essential, and identical results hold for any angular momentum state provided similar requirements of analyticity are satisfied.

We consider a single unstable particle of mass M which has two decay modes, both into two identical particles of mass m_r ($r=1,2$) with $2m_1<2m_2<M$. The resonance model gives, as scattering amplitude for two m_1 particles,

$$A_{11}(s)=ig_1^2/[s-M^2+\sum(a_r+ib_r)], \qquad (1)$$

where g_r is a coupling constant and

$a_r + ib_r$

$$= g_r^2\left(\frac{1-4m_r^2}{s}\right)^{1/2}\left\{\log\left[\frac{s^{1/2}+(s-4m_r^2)^{1/2}}{s^{1/2}-(s-4m_r^2)^{1/2}}\right]+i\pi\right\}.$$

If we choose both coupling constants suitably small, the amplitude A_{11} given by (1) (and similarly A_{12} and A_{22}) has a resonance for s near to M^2. This is due to a pole P on the third sheet, reached directly from the physical sheet by going down through the cut $4m_2^2 < s$. However, it is easily seen from (1) that there is also a pole P' to be found by encircling the threshold $s = 4m_2^2$ in the opposite direction; that is, by crossing the real axis in $4m_1^2 < s < 4m_2^2$. The precise location of the pole P' depends on the values of g_1 and g_2; if $g_1 > g_2$, P' lies on the second sheet (that is, the sheet directly accessible from the physical sheet through $4m_1^2 < s < 4m_2^2$), while if $g_2 \gg g_1$, P' is on the sheet reached from the second sheet by recrossing the real axis in $4m_2^2 < s$.

The pole P we call the dominant pole, and P' its shadow pole. Since both of these are accompanied by the usual complex conjugate poles Q and Q', we see that in this simple two-channel model there are, in all, four poles, the dominant pole P, its shadow pole P', and their conjugate poles Q and Q'.

Our simple model can be easily generalized to include n channels with masses $m_1, m_2, \cdots, m_n$. In this case the amplitudes have in all 2^n poles, which may be classified as a dominant pole P_1, say, its shadows $P_2, P_3, \cdots, P_N$ ($N = 2^{n-1}$), and their conjugate poles Q_1, Q_2, etc. The sheets on which these poles lie depend on the values of $g_1, \cdots, g_n$. If, for example, we consider the case $g_1 > \sum_2^N g_r$, then n of the N poles $P_1, P_2, \cdots$ lie on the n unphysical sheets directly accessible from the physical sheet; that is, the sheets reached directly through $4m_{r-1}^2 < s < 4m_r^2$.

In the general n-channel problem, we have shown from unitarity and analyticity that a resonance may be represented by anything from a single pole P, with its complex conjugate Q, to (2^{n-1}) poles P_i, each with a conjugate pole Q_i. As we show below, the behavior of the resonance multiplets of the scheme SU_3 when the symmetry is broken can be simply explained if the resonances not only have a dominant pole but also a series of shadow poles.

We consider next the experimental consequences of a wide resonance just above a threshold for a competing channel to which it is strongly coupled. In the neighborhood of the resonance the denominator of $A_{11}(s)$ is approximately of the form

$$(x^2 - c) + i(d + fx), \tag{2}$$

where x^2 is the energy relative to the threshold $4m_2^2$, and x is positive when x^2 is positive. It is readily seen from (2) that if f^2 is greater than $2c$, there will be a "false resonance" indicated by the cross-section peaking at threshold. Qualitatively, this corresponds to the width associated with the new channel being comparable with the distance above threshold. More generally in the absence of a false resonance there will be some distortion of the resonance shape.[5] This interference between a resonance pole and its shadow may occur in any partial wave; it is quite distinct from the well-known cusp or step effect which comes from a branch point in the numerator of the partial-wave S matrix.

Finally we consider the problem posed by Oakes and Yang, of how degeneracy of resonances in an SU_3 multiplet can develop continuously as symmetry-breaking interactions are switched off.[1] As Oakes and Yang indicate, there is no path by which a single resonance pole can emerge as a bound state without conflicting with any simple mass formula. However, if, as our work indicates, every resonance corresponds to a series of poles on different Riemann sheets, no such difficulties arise.[6] With each resonance multiplet in the unitary symmetry scheme we suggest there is associated a set of poles on the second sheet, together with sets of shadow poles on other Riemann sheets. The dominant resonance pole will not, in general, be the pole on the second sheet but will be the shadow pole on the sheet nearest to the experimental value of energy at resonance.

When the symmetry-breaking interaction is gradually switched off, the resonance poles will move towards their final degenerate positions. Their paths can be followed in our model by decreasing M^2 through real values in Eq. (1). As M^2 decreases past each threshold, the role of dominant pole is transferred from one shadow pole P_r, say, to the next P_{r-1} until finally the pole on the second sheet becomes dominant. Decreasing M^2 past the lowest threshold, this pole goes through the threshold branch point on to the physical sheet becoming a bound state. The shadow poles remain on lower sheets. If a resonance multiplet becomes a set of bound states of equal masses when full symmetry is established, it is interesting to note that the corresponding bound-state poles will not, in general, be the original set which were the dominant poles of the resonance multiplet. Thus it appears that experimental resonances do not necessarily arise directly from the poles of a symmetry multiplet but may arise instead from the shadows of these poles.

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†NATO Fellow.

[1]R. J. Oakes and C. N. Yang, Phys. Rev. Letters 11, 174 (1963).

[2]P. V. Landshoff, Nuovo Cimento 28, 123 (1963).

[3]A detailed account of these arguments will be given in a future paper.

[4]R. J. Eden, Proc. Roy. Soc. (London) A210, 388 (1952); 217, 390 (1953).

[5]We are indebted to P. V. Landshoff for discussions on the possibility of false resonances in connection with production thresholds.

[6]Since this Letter was first submitted, we have learned of related work by R. H. Dalitz and G. Rajasekharan, reported at the Siena Conference on Elementary Particles, Siena, Italy, 1963 (unpublished). We are indebted to Professor Dalitz for information about their approach to the Oakes-Yang problem.

QUANTUM ELECTRODYNAMICS*

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The unrenormalized Schwinger-Dyson equations[1,2] for the Green's functions of ordinary quantum electrodynamics have been examined in a systematic nonperturbative approximation scheme. We have obtained the following results. We can obtain finite solutions of the equations without infinite renormalizations if and only if the mechanical mass of the particle is taken to be zero. This still allows a finite physical mass for the particle. The asymptotic behavior of the finite, unrenormalized electron and photon propagators is found to be

$$1/S(p) \to \gamma p + O(p\beta), \tag{1}$$

as $p\to\infty$, where $\beta = -1 + (1-3\alpha_0/\pi)^{1/2}$, and

$$1/D(k^2) \to k^2\{1 + O[(k^2)\beta']\}, \tag{2}$$

as $k^2\to\infty$, where $\beta' = -\alpha_0/3\pi$ and where α_0 is the bare electron coupling constant. The second form holds only for small α_0. This behavior implies that all renormalization constants are finite. In outline, these results are obtained as follows.

If the unrenormalized Green's functions actually exist, then the following statements can be made. The electron Green's function $S(p)$ has the spectral representation

$$1/S(p) = \gamma p + m_0 - \int dK\, r(K)/(\gamma p + K), \tag{3}$$

where the integral over the mass spectrum converges; and therefore, as $p\to\infty$,

$$1/S(p) \to \gamma p + m_0 \tag{4}$$

for any reasonable $r(K)$. Secondly, because of Ward's identity,

$$k^\mu \Gamma_\mu(p+k,p) = S^{-1}(p+k) - S^{-1}(p), \tag{5}$$

we have for fixed k by differentiating (5) with respect to k as $p\to\infty$

$$\Gamma_\mu(p+k,p) \to \gamma_\mu \tag{6}$$

[actually faster than S approaches its asymptotic value because of the difference which appears in (5)]. Finally the photon Green's function has the spectral form

$$1/D(k^2) = k^2 + k^2\int d\lambda^2 s(\lambda^2)/(\lambda^2+k^2), \tag{7}$$

so

$$D(k^2) \to 1/k^2, \tag{8}$$

as $k^2\to\infty$ for any reasonable $s(\lambda^2)$. We again emphasize that all of these statements refer to unrenormalized Green's functions, none of which exists in perturbation theory. In ordinary language, the first statement implies that δm and Z_2 $(=Z_1)$ are finite. The final statement implies that Z_3 is finite.

The systematic approximation scheme used by