‡Boese predoctoral Fellow.

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POSITION OF RESONANCE POLES NEAR THE THRESHOLD OF A CHANNEL*

Marc Ross†
Indiana University, Bloomington, Indiana
and Lawrence Radiation Laboratory, University of California, Berkeley, California
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A recent Letter of Oakes and Yang[1] raises the question of the position of resonance poles of the S matrix in a multichannel situation when the resonance is near the threshold of one of the coupled channels. We will discuss this question stressing the well-known fact that more than one pole of S is usually associated with any resonance.[2] It is generally assumed that in the case of a narrow resonance there is just one "nearby" pole which can be reached by going a short distance from the physical line into the lower half k^2 plane. One readily finds, however, that there will usually be two nearby poles if the resonance lies very near a threshold. Furthermore, the nearby pole, in case the resonance lies above threshold, is not the "same pole" in case the interaction is strengthened or masses increased to that the "same resonance" lies below the threshold. These general considerations seem to preclude general use of a pole of S to define the position of a resonance. This is not an especially new result, but we feel it needs emphasis at the present time. In a second paper[3] we will consider the other customary definitions of resonance position and briefly reexamine the problem investigated by Oakes and Yang: Under what conditions can perturbation theory describe the motion (due to a change in interaction or channel thresholds) of a multichannel resonance?

We restrict ourselves to a finite number of two-body channels, and we neglect the possibility (which in practice can be very important) that the left-hand cuts in amplitudes coupling closed channels occur at or above the threshold in question. The threshold in question is generally above some thresholds and below others that play a significant role in the resonance. In terms of the T matrix for a single partial wave, unitarity dictates that

$$T^{-1*} - T = 2i\rho\theta, \qquad (1)$$

where T is symmetric by time-reversal invariance; we use the normalization $T_{jj} = W\exp[i\delta_j - (2l_j+1)\ln k_j]\sin\delta_j$; ρ is a diagonal matrix; $\rho_j = \exp[(2l_j+1)\ln k_j]/W$ and θ is diagonal with elements

$$\theta_j(W) = \begin{cases} 1 \text{ above the } i\text{th threshold} \\ 0 \text{ below the } i\text{th threshold.} \end{cases}$$

We first confine our attention to the <u>neighborhood</u> of the ith threshold, assuming for simplicity that the resonance width is small compared to the distance to other thresholds. We can thus stay on a single uncut sheet with respect to the variables k_j^2, $j \neq i$, since we include only the ith channel threshold branch point in our region. We chose to be on the sheet where $k_j > 0$ for all j below i, and $-ik_j > 0$ for all j above i (this sheet straddles both physical and unphysical sheets). All k_j^2's, $j \neq i$, are analytic functions of k_i^2 in our region. From (1) we write in our region[4]

$$T^{-1} = M(k_i^2) - i\rho, \qquad (2)$$

where $M(k_i^2) = M^*(k_i^{2*})$ and M is analytic in k_i and k_i^2 in the region. We neglect the possibility of isolated poles of M between the ith threshold and the resonance.

At resonance poles of T, $\det T^{-1} = 0$. We, therefore, have the eigenvalue equation

$$(M - i\rho)\zeta = \lambda\zeta, \qquad (3)$$

where $\lambda = 0$ at poles of T and ζ is the resonance eigenvector (determining the relative weight of various channels in the resonant system). We can write in our region that the pole(s) are at solutions of

$$\lambda = \lambda_c(k_i^2) - i\lambda_0(k_i^2) - i\lambda_i(k_i^2)k_i^{2l+1} = 0. \quad (4)$$

By explicit construction of λ in the two-channel case (see below), one readily sees that aside from left-hand cuts, λ is analytic except for the physical cut with branch points at the various thresholds and other cuts whose branch points are points where two solutions λ of (3) coincide. If there is only one resonance in the vicinity of the ith threshold, we can then conclude that λ_c, λ_0, and λ_i are analytic functions of k_i^2 in our region. By definition we take λ_c and λ_0 to be real analytic functions. Let

$$\rho = \rho_c + \rho_0 \rho_i,$$

where ρ_c includes all channels below i (the closed channels), ρ_0 all channels above i, and ρ_i is the diagonal matrix with all elements equal to zero except the ith which is equal to $\exp[(2l_i+1)\ln k_i]/W$. At the ith threshold,

$$\lambda_c = \zeta^\dagger(M - i\rho_c)\zeta/\zeta^\dagger\zeta, \quad \lambda_0 = \zeta^\dagger\rho_0\zeta/\zeta^\dagger\zeta,$$

so that

$$\lambda_0 \geqslant 0;$$

and

$$\lambda_i(k_i^2) = (\rho_i/k_i^3)\zeta_i^2/\tilde{\zeta}\zeta,$$

where ζ_i is the component of ζ in the ith channel and $\tilde{\zeta}$ denotes the transpose of ζ. The quantity $\zeta_i^2/\tilde{\zeta}\zeta$ is the reduced width in the ith channel. This quantity is approximately real and positive in case of a narrow isolated resonance. If the Breit-Wigner one-level formula applies (which seems to be true, for example, in the case of the baryon $\frac{3}{2}^+$ decuplet resonances), it is exactly real and positive on the real k_i^2 axis.

Corresponding to a narrow resonance near the ith threshold, we look for a solution of (4) near the physical region and near the threshold. We need to keep k_i^n dependence in (4) only for $n \leqslant 2l_i + 1$. We take $l_i = 1$ as typical of $l > 0$ cases, and write for (4)

$$a - bk_i^2 - i\beta - i\alpha k_i^3 = 0, \quad (5)$$

with a, b, β constant and real, $\beta > 0$, and α approximately real and positive. We have ignored here the k_i^2 dependence in λ_0 for simplicity since λ_0 should be small and positive in the whole neighborhood for a narrow resonance.

The parameters in (5) are further restricted by the condition that there be no nearby pole of T in the physical sheet (i.e., in the first quadrant in the k_i plane). A faraway solution of (5) in the physical sheet is not disturbing, as (5) is only valid in a small region. In fact, it is only in this way that we obtain a general prescription (for a variety of values of a) to avoid a pole in the physical sheet. This prescription follows by making a smooth transition from the elastic case ($\alpha > 0, \beta = 0$), letting β increase from zero: If we have a narrow resonance near threshold we find that b must be positive and relatively large,

$$b \gg |a^{1/3}|, |\beta^{1/3}|, \quad (6)$$

implying that one solution of (5) is

$$k_i \approx ib/\alpha. \quad (7)$$

This point may be in the physical sheet but lies far below threshold and merely simulates the left-hand cut in our approximate form for λ. We do not consider this solution further.[5]

The two other solutions of (5) are

$$k_i = \pm(a - i\beta/b)^{1/2} - (i\alpha/2b)(a - i\beta/b) + \cdots. \quad (8)$$

These are the positions of (the) two resonance poles in our region (i.e., if $|a|$ and β are small enough so there is a narrow resonance near threshold). In the limit of elastic scattering in channel i, $\beta = 0$, and the two poles are at points k_i and $-k_i^*$ for $a > 0$ as shown in Fig. 1. They merge if $a = 0$, and separate and go up and down the imaginary axis for $a < 0$. The pole on the positive imaginary axis in the latter case corresponds to a bound state. In the coupled-channel problem, (8) yields the pole positions shown in Figs. 2 and 3. We see that the pole positions are displaced relatively little from the elastic case, but that now the two poles never coincide for any value of a (no solution can generally be found for $\mathrm{Re}k_i = 0$).[6] Our result is then as follows: For $a > 0$ the resonance lies above threshold, and the nearby pole is just below the positive k_i axis. For $a < 0$ the resonance lies below threshold, and the pole close to the physical line is just left of the positive imaginary k_i axis. But these are not the same poles, as a is continuously changed to go from one situation to the other. Furthermore, if the resonance is very near threshold ($a \approx 0$) there are two poles near the physical line.

The situation for d or higher waves is essentially the same. For s waves it is more compli-

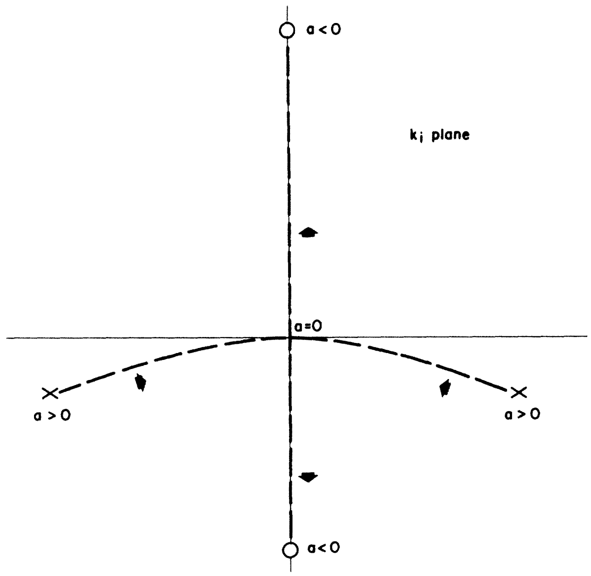

FIG. 1. Disposition of the two resonance poles associated with a resonance near the threshold of a p-wave channel i (we are on the sheet where momenta of other open channels are approximately positive and momenta of other closed channels are approximately positive imaginary). Here channel i is uncoupled to other open channels. Specific locations of the pairs of poles are shown. The dashed line indicates the motion of the poles as the interaction, through the parameter α, is strengthened.

cated and we discuss it only briefly. If the ith threshold is only weakly coupled to the resonance, then one finds that $b > 0$ and there are two poles disposed similarly to the $l = 1$ case. If the ith channel is strongly coupled to the resonance, then $b < 0$, we have a completely different situation and there is only one pole in our region near threshold instead of two.[7] The limiting case of this is elastic s-wave scattering where a single pole lies on the imaginary k axis below or above the origin depending on whether it corresponds to a bound state.

Finally, we look on other sheets with respect to branches associated with other channel thresholds. On each of the n sheets are roughly reflected the two poles as found above. For example, if there is a pole at k in the sheet considered above, there is a pole at $-k^*$ (all k_j's). The poles on these other sheets are, of course, all distant from the physical line near the ith threshold.

Certain points in the above argument are clarified by detailed consideration of the two-channel problem. We write, according to (2) for two p-

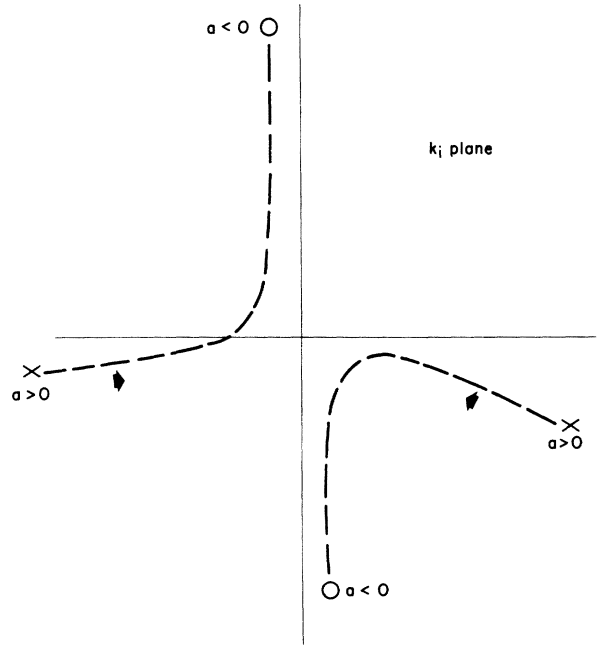

FIG. 2. The resonance poles in the presence of coupling to other open channels. See caption of Fig. 1.

wave channels,

$$T = \left[\begin{pmatrix} m_1 & m \\ m & m_2 \end{pmatrix} - i\begin{pmatrix} k_1^3 & 0 \\ 0 & k_2^3 \end{pmatrix}\right]^{-1},$$

where the m's are real analytic functions in the region of interest. We ignore the $1/W$ factor to simplify notation. The eigenvalues are

$$\lambda = \tfrac{1}{2}(m_1 + m_2 - ik_1 - ik_2)$$
$$\pm \tfrac{1}{2}[(m_1 - m_2 - ik_1 + ik_2)^2 + 4m^2]^{1/2}. \qquad (9)$$

One sees that in addition to the physical branches

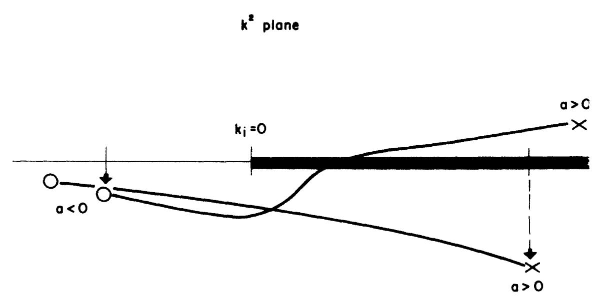

FIG. 3. The two resonance poles, as in Fig. 2, now shown in the k_i^2 plane. The arrow shows how nearby pole X is reached from the physical sheet if the resonance lies above threshold, and similarly when it lies below. The other pole in each case is reached by going around the threshold branch point.

starting at $k_1 = 0$ and $k_2 = 0$, there is another branch point at the point where the two solutions coincide. We assume this resonance coincidence point or double pole does not occur in the region of interest (i.e., only one resonance lies close to the threshold in question). Then λ will be an analytic function in the region except for the physical branch cuts. The other remarks concerning the division (4) of λ can also be confirmed by direct examination.

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†Present address: Physics Department, University of Michigan, Ann Arbor, Michigan.

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[3]Marc Ross, Phys. Rev. 131, 2678 (1963). These conditions are essentially the same as those needed in the present paper to carry through a discussion of the "normal" motion of resonance poles: (i) Large left-hand discontinuities in resonance amplitudes (coupling closed channels) do not occur near or above resonance. (ii) A second resonance in exactly the same partial wave does not occur in the relevant energy region. (iii) The shifts in momenta associated with motion of the resonance satisfy $|\Delta k|R \ll 1$, with R an appropriate range of forces. When we reconsider the application of Oakes and Yang, description of the observed lack of symmetry under SU_3 of the baryon decuplet resonances, we find that the first condition is probably the most serious. We conclude that although lowest order perturbation theory may yield a correct indication of mass shifts, perturbation theory should not be accurate in this case.

[4]See, for example, T. Fulton et al., Elementary Particle Physics and Field Theory (W. A. Benjamin, Inc., New York, 1962); W. Zimmerman, Nuovo Cimento 21, 249 (1961).

[5]One can turn the argument around and say that (6) indicates what we mean by "narrow" resonance and "near" threshold. Various effective range analyses suggest that near a narrow resonance b is of the order of the reciprocal range of forces (or the distance in momentum to the nearest large left-hand cut contributions). The relation (6) then states that we are examining a resonance whose appropriately defined width and distance from the ith threshold are small compared to the reciprocal range of forces.

[6]We note that there seems to be no objection in principle to other types of pole behavior, especially passage of the nearby pole through the origin or around the origin on the left, but this would be a special case requiring special behavior, particularly, of the quantity λ_0.

[7]See M. Ross and G. Shaw, Ann. Phys. (N.Y.) 13, 147 (1961), Appendix D.