packet C as shown in Fig. 1. But this variation in D carries with it slightly different g values for packet A as compared to C, and therefore different orbital hf fields, given by the following correspondence:

$$D \approx (\lambda/2)(g_{\parallel} - g_{\perp}) \rightarrow \delta D \approx (\lambda/2)(\delta g_{\parallel} - \delta g_{\perp}),$$

which, with $\delta g_{\perp} \approx -\frac{1}{2}\delta g_{\parallel}$, gives $\delta g_{\parallel} \approx (4/3\lambda)\delta D$, and from Eq. (2)

$$\delta H_{\text{orb}}^{\text{hf}} = 1.25\times 10^{5}(4/3\lambda)\delta D\langle 1/r^{3}\rangle_{\text{a.u.}} \text{ gauss}.$$

From A to C in Fig. 1 one has $\delta D = -\delta|D| = -200$ gauss, which together with the value of $\lambda \approx -280$ cm^{-1} leads to $\delta g_{\parallel} \approx +1.0\times 10^{-4}$ and $\delta H_{\text{orb}}^{\text{hf}} \approx +90$ gauss or $\delta H_{\text{total}}^{\text{hf}} = -90$ gauss, as compared to the observed value of -130 gauss. The agreement is quite satisfactory as other small effects may be present such as small variations in core polarization. Note that in going from A to C the increase of 130 gauss in the orbital hf field is out of a total orbital field of 170 000 gauss. One can anticipate the appearance of this phenomenon in other inhomogeneously broadened lines and correspondingly the probing of small changes in hf field with applied external electric field or axial pressure.

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BOUND STATES IN THE HEISENBERG FERROMAGNET*†

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This note reports the existence and some consequences of bound states in a Heisenberg ferromagnet.[1] In this work[2] the interactions of spin waves has been studied, following a method developed by Van Kranendonk.[3] A complete solution to the problem of two reverse spins in a crystal lattice of otherwise parallel atomic spins ($S = \frac{1}{2}$) has been found. A hard-core interaction has been introduced in the Hamiltonian to prevent these reversed spins from being localized on the same atomic site. As in the linear chain, it is found that for some eigenstates of this system, the two reversed spins are bound together in a stable complex, which travels through the crystal.

The Heisenberg Hamiltonian is

$$H = -2J\sum_{i<j}\vec{\mathbf{S}}_i\cdot\vec{\mathbf{S}}_j + \mu\sum S_i^{z} + \tfrac{1}{4}NcJ,$$

where the sum over i and j runs over all pairs of nearest neighbors in a simple cubic lattice, the spin vector $\vec{\mathbf{S}}_i$ being localized at the site $\vec{\mathbf{R}}_i$; $\mu = g\mu_B B$, where g is the g factor of the magnetic moment, μ_B the Bohr magneton, and B the externally applied magnetic field.

The space spanned by H can be divided into N orthogonal subspaces in which the total z component of the spin angular momentum is a constant. Each subspace is labeled by n, the number of reversed spins in the system.

The $n = 0$ subspace is the single ferromagnetic ground state; $n = 1$ has eigenfunctions which are the spin waves[4]; the $n = 2$ subspace was solved exactly and the results are described in this note. We consider spin deviation states $|i,j\rangle$ with two flipped spins at $\vec{\mathbf{R}}_i$ and $\vec{\mathbf{R}}_j$ and express the eigen-

functions in terms of these states:

$$\psi^{(2)} = \sum_{i,j} U(\vec{R}_i, \vec{R}_j)|i,j\rangle.$$

We obtain a set of difference equations already written by Van Kranendonk.[3] We introduce new coordinates, the center of mass of the two spin deviations and their relative separation, and transform the initial set of equations into a new set for $F_{\vec{K}}(\vec{R})$, where $\vec{R} = \vec{R}_i - \vec{R}_j$, and $\vec{K}$ is the total momentum of the two spin-deviations state

$$U(\vec{R}_i, \vec{R}_j) = \exp[\tfrac{1}{2}i\vec{K}\cdot(\vec{R}_i + \vec{R}_j)]F_{\vec{K}}(\vec{R}_i - \vec{R}_j).$$

This set reads

$$(E - 12J)F_{\vec{K}}(\vec{R}) + 2J\sum_{\vec{a}} F_{\vec{K}}(\vec{R}+\vec{a})\cos\tfrac{1}{2}\vec{K}\cdot\vec{a} = V(\vec{R})F_{\vec{K}}(\vec{R}),$$

where a represents the nearest neighbors, and $V(\vec{R}) = V_0\delta(\vec{R}) - 2J\sum_{\vec{a}}\delta(\vec{R}+\vec{a})$ which secures

$$U(\vec{R}_i, \vec{R}_j) = 0$$

required by our choice of $S = \frac{1}{2}$. This choice of a hard core was made by Van Kranendonk and held responsible for his erroneous results by Dyson.[5] $V(0)F_{\vec{K}}(0)$ is defined by

$$V(0)F_{\vec{K}}(0) = 2J\sum_{\vec{a}} F_{\vec{K}}(\vec{a})\cos\tfrac{1}{2}\vec{K}\cdot\vec{a};$$

in this way $F_K(0) \to 0$ as $V(0) \to \infty$ unless $E = 12 + V(0)$. Therefore the energy associated with the state $F_K(0)$ is infinite and this spurious unphysical state automatically disappears. To get rid of this solution Dyson introduced the "kinematical operator."

To solve the set we introduce Green's function[6]

$$K_{\vec{K}}(\vec{R}, \vec{R}') = -\frac{2}{N}\sum_{\vec{k}}\frac{\cos\vec{k}\cdot\vec{R}\cos\vec{k}\cdot\vec{R}'}{E - E(\vec{k}, \vec{K})},$$

where

$$E(\vec{k}, \vec{K}) = 12J - 2J\sum_{\vec{a}}\cos\tfrac{1}{2}\vec{K}\cdot\vec{a}\cos\vec{k}\cdot\vec{a},$$

and the set reads

$$F_K(\vec{R}) = 2J\sum\{K_{\vec{K}}(\vec{R}, 0)\cos\tfrac{1}{2}\vec{K}\cdot\vec{a} - K_{\vec{K}}(\vec{R}, \vec{a})\}F_K(\vec{a}).$$

We can always restrict $F_{\vec{K}}(\vec{R})$ to even functions in R and never have more than a set of three equations with three unknowns. Depending on whether E is inside or outside (below as it will turn out) of $E(\vec{k}, \vec{K})$, we have scattering or bound states. The energies of the bound states are obtained by equating to zero the determinant of the coefficients of the defining set.

For a given value of $\vec{K}$, the total momentum, the states can be classified according to the irreducible representation of the group of the wave vector $\vec{K}$. An integral representation of the Green's functions is easy to find in terms of a product of Bessel functions of order p of imaginary argument $I_p(t)$. For small $\vec{K}$ there is no bound state, therefore no bound states of very long wavelength. The fact that this set has always been handled in the limit $\vec{K}$ very small is responsible for the belief[5] that Bethe's[7] "spin complex" was characteristic of the one-dimensional chain. For an arbitrary $\vec{K}$, as $|\vec{K}|$ increases one bound state appears, then another, and finally a third one. To give an example, we choose $\vec{K}$ in the (1, 1, 1) direction. We have a Λ_1 solution[6] ($F_x = F_y = F_z$) and a doubly degenerate Λ_3 ($F_x = -F_y$; $F_z = 0$ and $F_x = F_y = -12F_z$). The energies of these states are, respectively, solutions of

$$\alpha = -(\mathcal{E}/\alpha + 3\alpha)\int_0^\infty dt\exp(\mathcal{E}t/\alpha)I,(t)I_0^2(t)$$

and

$$2\alpha = \int_0^\infty dt\exp(\mathcal{E}t/\alpha)I_0(t)\{I_0^2(t) + I_0(t)I_2(t) - 2I_1^2(t)\},$$

where

$$\alpha = \cos\tfrac{1}{2}\vec{K}\cdot\vec{a}$$

and

$$\mathcal{E} = E/4J - 3.$$

These integrals converge only if $\mathcal{E}/\alpha < -3$, which in turn means that the bound state is either below or at the bottom of the continium. The limiting case gives the value of $\vec{K}$ for which a bound state splits off below the continium. Λ_1 appears when $\cos\frac{1}{2}\vec{K}\cdot\vec{a} = 0.340$ and Λ_3 when $\cos\frac{1}{2}\vec{K}\cdot\vec{a} = 0.187$. Therefore these are spin-complex eigenstates. We note that the bound states are well separated from the bottom of the band. The gap is of $8J$ and they will give contributions of the order of $\exp(-2/\pi\theta)$ to any thermodynamical parameter and therefore do not appear in any series expansion in powers of θ. θ is the temperature in dimensionless units, $\theta = kT/4\pi J$.

A comparable analysis has been carried for the scattering and a general theory of spin-wave interaction using Van Kranendonk's method has been obtained and is hoped to be published later.

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HYPERFINE SEPARATION OF GROUND-STATE ATOMIC HYDROGEN*

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The hyperfine separation of hydrogen in its ground state, $\Delta\nu$(H), has been determined to the precision possible with currently available frequency standards by means of the hydrogen maser.[1-3] In this experiment two masers were operated simultaneously for purposes of tuning, checking internal consistency, and measuring the wall shift (the effect of collisions with the storage bulb on the transition frequency). A secondary frequency standard (National Company Model NC2001 cesium beam frequency standard) was also operated in the laboratory, and the frequency of this standard was monitored by one of the masers at half-hour intervals for a period of twenty-four hours. The average frequency of the secondary standard during this period was determined by J. A. Pierce of the Cruft Laboratory in terms of the 100-kc/sec time signal broadcast on Loran-C east-coast chain. The frequency of the Loran-C signal during the measurement period was subsequently determined by W. Markowitz and R. G. Hall of the U. S. Naval Observatory in terms of the weighted mean of a number of cesium-beam controlled frequency standards at different standards laboratories. By this method the maser frequency was referred to the mean of a number of primary standards, allowing a precision characteristic of the agreement among those standards.

Tuning a hydrogen maser involves tuning the cavity and adjusting the magnetic field to a known value. Cavity tuning was accomplished using the well-known fact that the maser oscillator frequency is "pulled" by a mistuned cavity by an amount proportional to the atomic resonance width.[3] By increasing the beam flux to allow spin-exchange collisions, the resonance width could be increased for this purpose by as much as a factor of four. The cavity was adjusted until the oscillation frequency was independent of the flux.

In addition to causing relaxation, spin-exchange collisions can introduce a small frequency shift.[4] This shift depends upon the atomic resonance width in just the same fashion as the cavity pulling, so by tuning the cavity with the above method any systematic frequency error is exactly canceled by compensating mistuning of the cavity. A preliminary experiment in which the resonance was broadened by deuterium confirms this result to well within the accuracy of concern here, and further experiments are in progress.

The magnetic field was adjusted for a Zeeman frequency of 10 kc/sec by the double resonance technique.[2] A 10-kc/sec signal from the frequency standard was introduced into the cavity, and the field could be trimmed to within 1 cps of resonance, which introduced negligible error in the frequency of the field-independent transition $(F=1, m=0) \rightarrow (F=0, m=0)$. By these means the masers could be independently and reproducibly tuned to agree to better than 1 part in 10^{12}. Fluctuations between the masers in ten-second counting periods were typically 1 or 2 parts in 10^{13}.

The following scheme was used for monitoring the local standard: The 5-Mc/sec output of the