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CORRECTIONS TO THE PAIRING INTERACTION IN SUPERCONDUCTIVITY*

E. M. Henley and L. Wilets
University of Washington, Seattle, Washington
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The theory of superconductivity expounded by Bardeen, Cooper, and Schrieffer[1] is based on a variational wave function involving time-reversed pairs. An alternative formulation given by Bogoliubov[2] is based on the principle of the compensation of dangerous diagrams, following a canonical transformation. The complete statement of the compensation principle can be shown to yield the generalized integral equation for the energy-gap parameter

$$\Delta_{\vec{k}} = -\tfrac{1}{2}\sum\nolimits_{k'} G_{\vec{k}\vec{k}'}\Delta_{\vec{k}'}/(\Delta_{\vec{k}'}^{\,2} + \epsilon_{\vec{k}'}^{\,2})^{1/2}, \tag{1}$$

where the $\epsilon_{\vec{k}}$ are "renormalized" normal single-particle energies, expressed relative to the Fermi energy, and $G_{\vec{k}\vec{k}'}$ is an "effective" pair interaction. To second order in the interaction V between pairs (after the canonical transformation has been carried out), G is given by[3]

$$G_{\vec{k}\vec{k}'} = \langle\vec{k}, -\vec{k}\,|V|\,\vec{k}', -\vec{k}'\rangle + 2\sum_{\vec{q}\vec{q}'}\frac{u_{\vec{q}}^{\,2}v_{\vec{q}'}^{\,2}\langle -\vec{k}\vec{q}'|V|-\vec{q}\vec{k}'\rangle\langle -\vec{q}k|V|-k'\vec{q}'\rangle - (uv)_{\vec{q}}(uv)_{\vec{q}'}|\langle -\vec{k}\vec{q}|V|-\vec{q}'k'\rangle|^2}{\xi_{\vec{k}} + \xi_{\vec{k}'} + \xi_{\vec{q}} + \xi_{\vec{q}'}}, \tag{2}$$

where, for simplicity, we have here assumed that the potential V acts only in singlet states. The energy $\xi_{\vec{k}}$ is $(\epsilon_{\vec{k}}^{\,2} + \Delta_{\vec{k}}^{\,2})^{1/2}$, and the u^2 and v^2 are given by

$$\begin{pmatrix} u_{\vec{k}}^{\,2} \\ v_k^{\,2} \end{pmatrix} = \frac{1}{2}\left(1 \pm \frac{\epsilon_{\vec{k}}}{\xi_{\vec{k}}}\right). \tag{3}$$

The second-order term in Eq. (2) proportional to $(u^2)(v^2)$ dominates over the $(uv)(uv)$ term and corresponds to intermediate scattering of a particle-hole pair. We note that for an attractive interaction (negative V) the second-order correction is effectively repulsive and tends to reduce the energy gap.

Bogoliubov, Tolmachev, and Shirkov[2] have argued that the ratio of the second-order terms in G to the first-order one are proportional to the energy gap, and are thus negligible. We have investigated this point in some detail, and find instead that this ratio is proportional to the strength of the interaction in the limit of small gaps. The second-order term can, in fact, be numerically very significant and in some instances no solution of Eq. (1) with Eq. (2) obtains.

We wish to draw a distinction between solving the integral equation (1) for the energy gap and an alternative perturbation procedure.[4] In the latter case one introduces a canonical transformation, the parameters of which (i.e., $\Delta_{\vec{k}}^{\,0}$) are determined by solving the lowest order (BCS) integral equation. The quasiparticle energies can then be evaluated by a perturbation treatment of the transformed Hamiltonian. To second order, the quasiparticle energy at the Fermi surface,[5] Δ^1, is given by the right-hand side of Eq. (1) except that $\Delta_{\vec{k}}$ is everywhere replaced by $\Delta_{\vec{k}}^0$. The two approaches can give quite different energy gaps and the differences do not, in general, approach each other even when the higher order terms included above are small. We believe that a finite-order perturbation procedure for obtaining the gap is not a proper method. The reason for this is that Eq. (1) is a homogeneous (nonlinear) integral equation which is not of the Fredholm type.

Solutions to Eq. (1) through second order in G have been investigated for a variety of potentials. For some ranges of parameters, no solution for $\Delta_{\vec{k}}$ obtains. However, the integral equation to second order can be written formally as

$$\Delta = fI_1(\Delta) - f^2I_2(\Delta) = fI_1[1 - fI_2/I_1] \approx fI_1/(1 + fI_2/I_1), \quad (4)$$

where $-f$ is the strength of the interaction. The final form of Eq. (4) is suggested by an infinite set of "particle-hole" diagrams if the potential is separable in the form

$$\langle \vec{k}_4\vec{k}_3 | V | \vec{k}_2\vec{k}_1 \rangle = v(\vec{k}_3, \vec{k}_2)v'(\vec{k}_4, \vec{k}_1) \quad (5)$$

which is valid for the exchange part of a local interaction.

The integral I_1 in Eq. (4) was obtained analytically or from previous calculations.[6] For evaluating I_2 we assumed the same $\vec{k}$ dependence for $\Delta_{\vec{k}}$ as for $\Delta_{\vec{k}}^0$ and carried out the integral numerically by a Monte-Carlo technique. In general, the error was less than 10%. Results for two of the three potentials investigated are displayed in Fig. 1, where the ratios of Δ/Δ^0 are plotted against the dimensionless BCS gap parameter. An effective-mass approximation was assumed for the normal particle energies, $\epsilon_{\vec{k}} = \hbar^2(k^2 - k_f^2)/2m^*$. The potentials studied were as follows:

Type I.

$$\langle \vec{k}_4\vec{k}_3 | V | \vec{k}_2\vec{k}_1 \rangle = -f\delta(\vec{k}_4 + \vec{k}_3 - \vec{k}_2 - \vec{k}_1) \text{ for } (k_i^2 - k_F^2)^2 < w^2,$$
$$= 0 \qquad \text{otherwise.} \quad (6)$$

As expected from phase-space considerations, the correction to Δ^0 is small for this interaction. We found, furthermore, that Δ/Δ^0 is insensitive to w and almost independent of Δ^0 or the strength of the interaction. The results are not graphed, but can be summarized as follows: For a range of $\Delta^0/(\hbar^2k_F^2/2m^*)$ from 10^{-4} to 5×10^{-2}, one obtains $\Delta/\Delta^0 \approx 0.80$ if $w/k_F^2 = 0.25$ and $\Delta/\Delta^0 \approx 0.85$

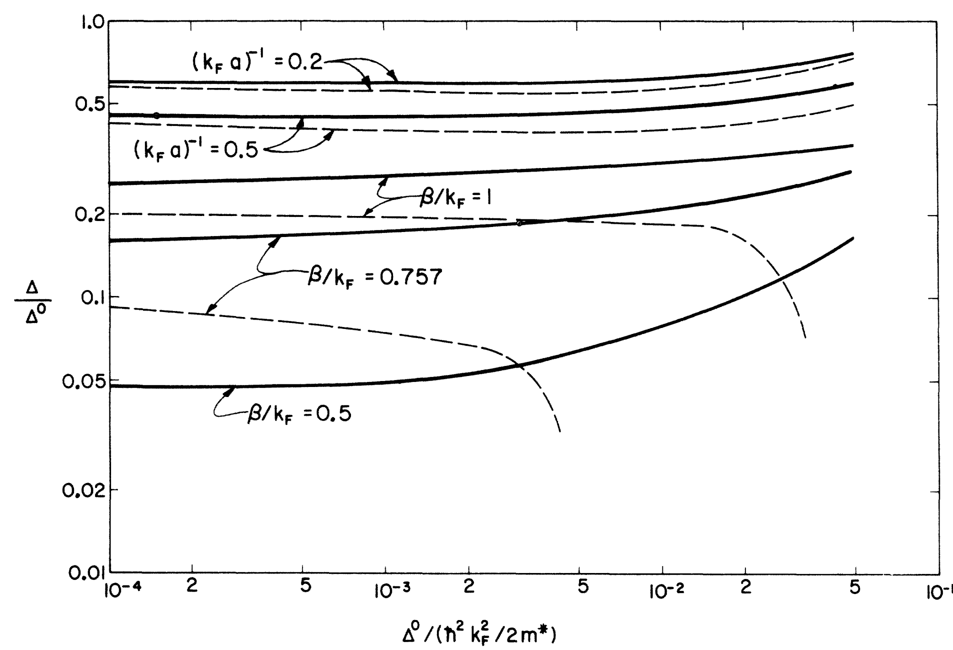

FIG. 1. Plot of the ratio of the "second-order" energy-gap parameter Δ to the first-order one Δ^0 as a function of Δ^0. The solid curves correspond to a solution with the last form of Eq. (4), the dashed ones to the first form of Eq. (4).

if $w/k_F^2 = 0.1$. We do not believe that type-I potential is realistic but have considered it because it reduces to the pairing interaction introduced by Bardeen, Cooper, and Schrieffer.[1]

Type II. This interaction is a nonseparable Gaussian interaction,

$$\langle \vec{k}_4\vec{k}_3 | V | \vec{k}_2\vec{k}_1\rangle = -\tfrac{1}{2}f\{\exp[-\tfrac{1}{2}a^2(\vec{k}_4-\vec{k}_2)^2] + \exp[-\tfrac{1}{2}a^2(\vec{k}_4-\vec{k}_1)^2]\}\delta(\vec{k}_4+\vec{k}_3-\vec{k}_2-\vec{k}_1), \quad (7)$$

a one-dimensional version of which we considered earlier[6] and is generalized here to obtain $\Delta_{\vec{k}}^0$. (In the notation of reference 6, we now let $\eta_k = k\Delta_{\vec{k}}^0/k_F\Delta^0$ instead of Δ_k^0/Δ^0; then calculations reported in reference 6 may be used directly.) Results are presented in Fig. 1 for $(k_F a)^{-1} = \frac{1}{2}$ and $\frac{1}{5}$. The solid curves correspond to the last form of Eq. (4); the dashed curves to the first form.

Type III. The last potential studied is a separable one of the Yamaguchi form,

$$\langle \vec{k}_4\vec{k}_3 | V | \vec{k}_2\vec{k}_1\rangle = -f\frac{\delta(\vec{k}_4+\vec{k}_3-\vec{k}_2-\vec{k}_1)}{[\frac{1}{4}(\vec{k}_2-\vec{k}_1)^2+\beta^2][\frac{1}{4}(\vec{k}_4-\vec{k}_3)^2+\beta^2]}. \quad (8)$$

In this case the second-order corrections are so large that no solutions with the first form of Eq. (4) exist if $\beta/k_F \lesssim 0.5$ nor for large Δ^0 if $\beta/k_F \gtrsim 0.5$. The dashed and solid curves of Fig. 1 have the same meaning as for case II.

In conclusion, we note that corrections to the BCS pairing interaction can be important to the evaluation of the energy gap. Furthermore, these corrections should be included self-consistently in the determination of the Bogoliubov canonical transformation. The correction terms tend to reduce the gap when the interaction is attractive. Further details and application to nuclear matter will be reported in a forthcoming publication.

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EMPIRICAL CHARACTERIZATION OF LOW-TEMPERATURE MAGNETORESISTANCE EFFECTS IN HEAVILY DOPED Ge AND Si

H. Roth, W. D. Straub, and W. Bernard
Raytheon Research Division, Waltham, Massachusetts

and

J. E. Mulhern, Jr.
Physics Department, University of New Hampshire, Durham, New Hampshire
(Received 8 July 1963)

Negative magnetoresistance has been previously observed at liquid helium temperatures by Sasaki, Yamanouchi, and Hatoyama[1] and by Furukawa[2] in heavily doped n-type germanium, and in both types of indium antimonide. In this Letter we report the existence of a similar, but positive, anomalous magnetoresistive effect in p-type germanium. We have also observed corresponding negative and positive anomalous magnetoresistance in heavily doped samples of n- and p-type silicon, respectively. An analysis of the magnetic field dependence of these phenomena over concentrations ranging from those of impurity-band conduction to extreme degeneracy suggests that the positive and negative effects are different manifestations of the same physical mechanism.

Samples were obtained from a series of heavily doped single crystals containing as impurities