RELATION BETWEEN THE WIDTH OF THE OMEGA AND THE REACTION $\pi+\pi\to\pi+\omega$†

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We propose a model for relating the reaction

$$\pi^+ + \pi^0 \to \pi^+ + \omega \tag{1}$$

to the decay of the omega into three pions,

$$\omega \to \pi^+ + \pi^- + \pi^0. \tag{2}$$

Our model relies on the dominance of the ρ-meson intermediate state; i.e., in Reaction (1), $\pi+\pi\to\rho\to\pi+\omega$; and in Reaction (2), $\omega\to\pi+\rho$ followed by $\rho\to 2\pi$. This model arises as the result of an S-matrix approach if one makes the usual approximation of considering only nearby singularities; it has been proposed for ω decay by Gell-Mann, Sharp, and Wagner[1] and by Frazer and Wong,[2] and for relating Reactions (1) and (2) by Sakurai.[3] The principal difference between our model and Sakurai's is that ours is derived from consideration of the unitarity and analyticity of the S matrix, and remains within the bounds imposed by unitarity. Sakurai's model, based on a vector-meson field approach, neglects the threshold singularity in Reaction (1). Consequently, it is impossible to fit the data of the preceding Letter[4] with this model without violating the unitarity limit around 1.5 BeV or below.

We obtain a formula for Reaction (1) by first considering the partial-wave scattering amplitudes. Conservation of isotopic spin and the exclusion principle allow only $I=1$ and odd total angular momenta. We therefore focus our attention on the $J=I=1$ state, which should be dominant over most of the energy range of the experiment in the preceding Letter. In order to construct a partial-wave amplitude for Reaction (1), we must simultaneously consider the processes

$$\pi+\pi\to\pi+\pi, \tag{3}$$

$$\pi+\omega\to\pi+\omega. \tag{4}$$

The amplitudes for these processes in the $J=I=1$ state will be designated $M_{11}(s)$ and $M_{22}(s)$, respectively, with $M_{12}(s)$ for Reaction (1). The M_{ij} are normalized to remove threshold factors: $M_{ij} = s^{1/2}T_{ij}/(q_iq_j)^{3/2}$, where q_1 and q_2 are center-of-mass momenta in the two-pion and π-ω states, respectively; and where the T_{ij} are the usual T-matrix elements, normalized such that below the π-ω threshold, $T_{11}=e^{i\delta}\sin\delta$.

To obtain a unitary scattering amplitude, we use the well-known method[5] of setting $M=ND^{-1}$, where all quantities are matrices, and where the left-hand cut, which is determined by the nature of the interactions which one chooses to include, is contained in the matrix $N_{ij}(s)$. We shall not attempt to calculate the interaction, but rather to represent it in terms of a few parameters in order to obtain an "effective-range" formula.[6] We shall then use this formula to extrapolate the amplitude $M_{12}(s)$ from the neighborhood of the ρ-meson pole to the physical region for Reaction (1). Thus the present calculation is the reverse of the Dalitz-Tuan analysis in which K-N scattering data were used to predict a resonance below the threshold.

In a one-channel calculation if one represents the left cut by a simple pole, one obtains the usual effective-range formula. We follow the same procedure, setting $N_{ij}(s)=n_{ij}/(s+s_0)$. The algebra can now be simplified if one exploits the free multiplicative factor in N and D; i.e., $M=ND^{-1} = N'D'^{-1}$, where $N'=NL$ and $D'=DL$, and L is an arbitrary matrix. We choose $L=n^{-1}$, so that

$$N_{ij}' = \delta_{ij}/(s+s_0). \tag{5}$$

Hereafter we shall drop the primes. Then it follows in the usual manner that

$$D_{ij}(s) = c_{ij} - \delta_{ij}(s-m_\rho^2)K_i(s), \tag{6}$$

where the c_{ij} are arbitrary real constants, with $c_{12}=c_{21}$ to make the T matrix symmetric. For convenience we have made a subtraction at the square of the ρ-meson mass, so that

$$K_i(s) = \frac{1}{\pi}\int_{s_i}^{\infty} ds' \frac{q_i^{\,3}(s')}{s'^{1/2}(s'-s)(s'-m_\rho^2)(s'+s_0)}, \tag{7}$$

where $s_1 = 4m_\pi^2$ and $s_2=(m_\pi+m_\omega)^2$. These equations determine the matrix M in terms of four parameters. We eliminate two of these by requiring a zero of the determinant of D to represent a ρ meson with the correct position and width. The position is fixed at m_ρ^2 by requiring that $\mathrm{Re}[\det D(m_\rho^2)]=0$, which results in the condition $c_{11}c_{22}=(c_{12})^2$. In order to adjust the ρ-meson width, we examine the form of the T matrix implied by Eqs. (5) and (6) in the vicinity of m_ρ^2.

Expanding about this point, we obtain the Breit-Wigner form

$$M_{ij}(s)=\frac{(\gamma_i\gamma_j)^{1/2}}{m_\rho^{\,2}-s-i\gamma_1q_1^{\,3}/s^{1/2}}, \tag{8}$$

where the partial widths γ_1 and γ_2 are given by

$$\gamma_1=\frac{c_{22}}{s_0[c_{11}K_2(m_\rho^{\,2})+c_{22}K_1(m_\rho^{\,2})]};\ \gamma_2=\gamma_1c_{11}/c_{22}. \tag{9}$$

Now γ_1 is adjusted to give the observed ρ-meson width, whereas γ_2 is directly related to the coupling constant at the ρ-π-ω vertex.[7,8] We have now reduced the number of parameters in our effective-range formula for Reaction (1) to two; we choose these to be γ_2 and α, where $\alpha\equiv s_0c_{22}$. We can now calculate the cross section for $\pi^+ +\pi^0\to\pi^+ +\omega$ via Eqs. (5) and (6) and the relation

$$\sigma_{12}=12\pi\lambda^2|T_{12}|^2. \tag{10}$$

An example is shown in the solid curve in Fig. 1 for $\gamma_2=2.5$ and $\alpha=5$.[9] This curve corresponds to a resonance, i.e., to a pole in the T matrix close to the physical region. The parameter α controls the position of this resonance, whereas its width and height are controlled by γ_2.[10]

In the preceding Letter several curves were given which fit the data on Reaction (1). All of these implied the existence of a second resonance with the quantum numbers of the ρ meson at a mass of about 1200 MeV, with a width of 200-400 MeV, which we shall refer to as a ρ'. Our effective-range formula is unable to fit the data without such a resonance. For a given value of the parameters we can calculate the corresponding cross section for $\pi+\pi\to\pi+\pi$. In Fig. 2 this cross section is shown for the same values of the parameters as in Fig. 1, which correspond to one of the fits given in the preceding Letter. There is no experimental evidence for a ρ' in double-pion production experiments,[11] but a bump as low and broad as the second one in Fig. 2 might be difficult to detect. It should also be present in the $I=1$ $K\bar{K}$ channel, but we cannot calculate the partial width for $\rho'\to K\bar{K}$ since we have neglected the $K\bar{K}$ channel in our calculation. A rough estimate can be made by using unitary symmetry corrected by the ratio of $\pi\pi$ and $K\bar{K}$ phase spaces. We estimate in this manner that the ρ' should appear in $\pi+\pi\to K+\bar{K}$, $I=1$, as a peak about 2 mb high.[12]

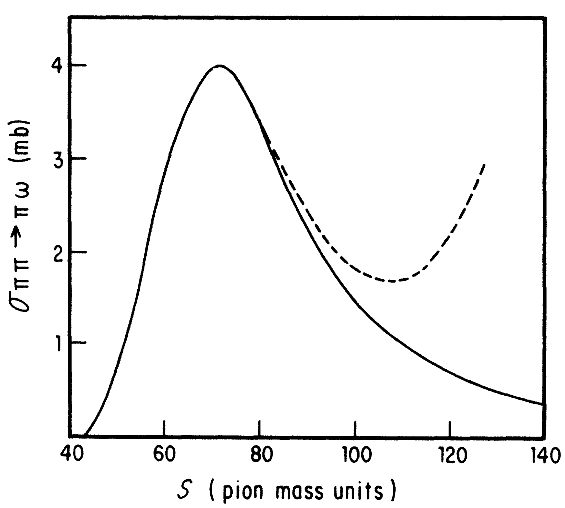

FIG. 1. The cross section for $\pi^+ +\pi^0\to\pi^+ +\omega$ for $\alpha=5$, $\gamma_2=2.5$. The solid curve is the $J=1$ state; the dashed curve includes in addition an estimate of the higher waves.

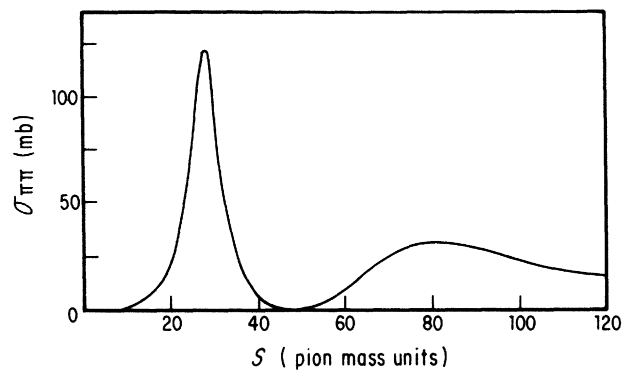

FIG. 2. The pion-pion elastic cross section for the same values of the parameters as used in Fig. 1.

The dashed curve in Fig. 1 represents an estimate of the contribution of the partial waves with $J\geqslant 3$. This has been obtained by including the effect of the ρ-meson pole in the t and u channels; $M_{12}(s)$ is replaced by

$$M_{12}(s)+M_{12}'(t)+M_{12}'(u). \tag{11}$$

The primes signify that we have first subtracted out the pole at $s=-s_0$, and then removed the $J=1$ portion in the s channel, since this is already included in $M_{12}(s)$. The dashed curve is, of course, not to be taken seriously when it becomes much larger than the solid one. No attempt has been made to take account of the unitarity condition for the $J\geqslant 3$ waves. However, the dashed curve gives us an indication of the energy at which these higher waves will make a significant correction to the model developed above, and thus gives an upper limit on the energy at which the model should be used.

Let us now turn to the calculation of the width of the omega, Reaction (2). By using crossing symmetry we can express this width directly in terms of the amplitude M_{12} for Reaction (1) integrated over the phase space of the three pions:

$$\Gamma = \frac{3}{4\pi}\int_{4m_\pi^2}^{(m_\omega - m_\pi)^2} ds \left(\frac{q_1 q_2}{m_\omega}\right)^3 \times \int_{-1}^{1} d(\cos\theta)\,\sin^2\theta\,|f(s,\cos\theta)|^2, \quad (12a)$$

where

$$f(s,\cos\theta) = \overline{M}_{12}(s) + \overline{M}_{12}(t) + \overline{M}_{12}(u), \quad (12b)$$

and where

$$t = 2m_\pi^2 - 2\omega_1\omega_2 + 2q_1q_2\cos\theta,$$

$$u = 2m_\pi^2 - 2\omega_1\omega_2 - 2q_1q_2\cos\theta,$$

$$\omega_i = (q_i^2 + m_\pi^2)^{1/2}.$$

The symbol $\overline{M}_{12}$ denotes that the interaction pole at $s = -s_0$ has been subtracted out. This pole represents the effect of the left-hand, or "interaction," singularities, the nearby part of which is given by just the last two terms in Eq. (12b). To avoid counting these terms more than once, we remove the interaction pole. The result of numerical integration of Eq. (12) has been given in the preceding Letter for the curves used.

Another method of calculating the ω width would be to take the value of $f_{\rho\pi\omega}$ implied by a given value of γ_2 and insert this into the Gell-Mann, Sharp, and Wagner (GMSW) formula.[1] The results are somewhat different; for example, for $\gamma_2 = 4.1$ and $\alpha = 3.6$ we obtain $\Gamma_\omega = 10.7$ MeV, whereas the GMSW formula gives 9.7 MeV. The difference comes about because (a) we have used the functional form of M_{12} derived from our S-matrix approach, whereas GMSW have used conventional perturbation-theoretic propagators, and (b) our formula includes the effect of the ρ' pole. It is reassuring that the results of the two approaches agree as closely as they do.

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[7]We have used 110 MeV as the ρ-meson width. Changes in this value will be reflected as the same percent change in the calculated omega width.

[8]If the factor at the ρ-π-ω vertex is

$$(f_{\omega\pi\rho}/m)\epsilon_{\mu\nu\lambda\sigma}\epsilon_\mu^{(\omega)}k_\nu^{(\omega)}\epsilon_\lambda^{(\rho)}k_\sigma^{(\rho)},$$

then the relation between γ_2 and $f_{\omega\pi\rho}$ is

$$f_{\omega\pi\rho}^2/4\pi m^2 = 3\gamma_2/m_\rho^2.$$

[9]We use units in which the mass of the charged pion is set equal to unity.

[10]This is reminiscent of the Chew-Low theory, in which the width of the (3,3) resonance is determined by the pion-nucleon coupling constant.

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