# Entanglement Entropy for the Long-Range Ising Chain in a Transverse Field

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We consider the Ising model in a transverse field with long-range antiferromagnetic interactions that decay as a power law with their distance. We study both the phase diagram and the entanglement properties as a function of the exponent of the interaction. The phase diagram can be used as a guide for future experiments with trapped ions. We find two gapped phases, one dominated by the transverse field, exhibiting quasi-long-range order, and one dominated by the long-range interaction, with long-range Néel ordered ground states. We determine the location of the quantum critical points separating those two phases. We determine their critical exponents and central charges. In the phase with quasi-long-range order the ground states exhibit exotic corrections to the area law for the entanglement entropy coexisting with gapped entanglement spectra.

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Long-range (LR) interactions have attracted a lot of attention since they could produce interesting new phenomena [1–4]. Recently there have been impressive advances in controlling experimentally quantum systems. In particular it has been shown by Britton *et al.* [5] that LR Ising anti-ferromagnetic interactions can be induced in trapped beryllium ions. This is only the most recent of a series of experimental results on using trapped ions to simulate spin models [6–8]. Motivated by these results we analyze the phase diagram of the antiferromagnetic LR Ising Hamiltonian in the presence of a transverse field (LITF). The difference with the standard Ising model in a transverse field (ITF) is that the two-body interactions extend to all spins and decays with their distance $r$ as $r^{-\alpha}$ with $\alpha > 0$.

For the LITF we (i) the determine the full phase diagram as a function of $\alpha$ (ii) quantify the increase of complexity induced by the LR interaction for the classical simulation of the model and (iii) characterize the phase transitions.

Regarding both (i) and (iii), we identify two different phases. One of them, dominated by the local part of the Hamiltonian, is gapped and presents patterns of quasi-long-range order (QLRO) induced by the LR part of the Hamiltonian. This is exotic, since normally QLRO is associated to gapless phases. The other, dominated by the LR terms of the Hamiltonian, presents antiferromagnetic LR order (LRO) in the form of Néel ground states. Between them, we observe a line of quantum phase transitions, whose nature depends on the value of $\alpha$. They either are in the same universality class than the ITF ($\alpha > 2.25$), or present new universal behaviors (for $\alpha \leq 2.25$).

Concerning (ii), we focus on the entanglement entropy of the ground states of the LITF as a function of both $\alpha$ and system size. A common belief, (see, however, Ref. [9] for an updated perspective) relates the amount of entanglement in a state with its classical simulability [10,11]. This translates into the fact that those states that obey the "area law" for the entanglement, can be simulated classically. For example, all ground states of gapped short range (SR) Hamiltonians obey the "area law" [12,13]. Here we show that the ground states of the LITF can violate it. The violations are at most logarithmic in the system size so that the ground state of the LITF can still be approximated efficiently with matrix product states (MPS) [14].

Our studies complement the existing one in several ways. On one side most of the literature has focused on LR dipolar interactions decaying with the distance as $r^{-3}$ [15–20]. Much less has been done for a generic LR interactions of the type $r^{-\alpha}$ as the one we consider here [21–23]. For these systems, even less has been done with respect to the interplay between antiferromagnetism and LR interactions [24,25] (in the context of classical spin-glasses people have worked on related LR models, see for example Ref. [26] and references therein).

From the point of view of the complexity of the ground states, Eisert and Osborne have constructed, in the context of fermions interacting through a unshielded Coulomb potential ($\alpha = 1$), a specific example of a long-range gapped Hamiltonian with logarithmic corrections to the entanglement entropy in Ref. [27]. Here we find that the phenomenon is more general. For the LITF the corrections are there not only at $\alpha = 1$ but also for arbitrary $\alpha \leq 2$ (surprisingly, we do not find any stronger corrections for the small $\alpha$s). In the same phase, for $\alpha > 2$ the corrections are sub-logarithmic. Finally, we also identify a phase with no corrections at any $\alpha$. The corrections coincide with not only a gapped Hamiltonian but also with a gapped entanglement spectrum.

As a side result, we have improved current MPS techniques by generalizing the time dependent variational principle (TDVP) [28,29] to LR interactions (alternative approaches are available [16,17,19,20,30,31]). The generalization is described in the Supplemental Material [32].

*The model.*—We study a one dimensional spin chain with open boundary conditions. We analyze the ground state of the system described by the LITF Hamiltonian

$$H(\theta, \alpha) = \sin(\theta)\sum_{i,j} \frac{1}{|i-j|^{\alpha}} \sigma_x^i \sigma_x^j + \cos(\theta)\sum_i \sigma_z^i, \quad (1)$$

where $i$, $j$ are two arbitrary points of the 1D chain, $\alpha \geq 0$. We consider the antiferromagnetic regime, $0 \leq \theta \leq \frac{\pi}{2}$. This is the interesting regime for the experimental results in Ref. [5] and we are interested in studying the interplay between LRO and frustration; it is worth remarking that the ferromagnetic LITF has been already studied elsewhere [1,2,22].

The phase diagram is obtained through the entanglement entropy of half system defined as

$$S_{L/2} = -\mathrm{tr}\rho_{L/2} \log\rho_{L/2}, \quad (2)$$

where $\rho_{L/2} = \mathrm{tr}_{i_1 \cdots i_{L/2}} |\Omega\rangle\langle\Omega|$ and $|\Omega\rangle$ is the ground state of the system. We use the maxima of $S_{L/2}$ as the signature for the phase transitions.

We then analyze the entanglement spectrum (ES) defined in terms of the logarithm of the reduced density matrix

$$h_i = \log(\rho_i), \quad (3)$$

where $\rho_i$ are the eigenvalues of $\rho_{L/2}$. For $\theta > \theta_c$ the ES can be fully described by using perturbation theory (PT). For $\theta < \theta_c$ we observe both a perturbative and a nonperturbative regime depending on the value of $\alpha$.

At the critical point we consider the finite size scaling of the correlation functions

$$\langle \sigma_x^{L/2} \sigma_x^{L/2+L/5} \rangle \propto L^{-2\Delta_x}, \qquad \langle \sigma_z^{L/2} \sigma_z^{L/2+L/5} \rangle \propto L^{-2\Delta_z}. \quad (4)$$

The corresponding exponents, as a function of $\alpha$ define the universality class and present two different regimes. A SR regime, where the critical exponents are the ones of the ITF, and a LR regime, where the exponents vary continuously with $\alpha$.

*Numerical results.*—We use MPS techniques [33] through a variational algorithm (known as TDVP [28,29]) to obtain the best possible MPS for the ground state of 1. In order to deal with the LR, the Hamiltonian is encoded in a matrix product operator [33] in a way to correctly reproduce the desired power law $r^{-\alpha}$ in the range of distances $1 \leq r \leq L/2$ [34]. This requires an extension of the original TDVP algorithm (described in the Supplemental Material [32]). We have performed simulations of finite chains with length $L$ in the range $20 < L < 150$ and open boundary conditions. For each simulation, we have increased the MPS bond dimension $\chi$ up to convergence using the criterion described described in the Supplemental Material [32]. This typically happens at values of $\chi \leq 100$.

*Phase diagram.*—In the antiferromagnetic case for all values of $\alpha > 0$ the system shows two phases. For values of $\theta \sim 0$, $\theta \ll \theta_c(\alpha)$, the ground state $|\Omega\rangle$ can be understood as a perturbative modification of the product state locally pointing along $z$. This is a gapped phase, where elementary excitations are spin-flips. For values of $\alpha \leq 1$ and $\theta \ll \theta_c(\alpha)$, the ground state starts to encode patterns LR of correlations that suggest the existence of a nonperturbative regime (see Fig 3, right panel). However, when passing from the perturbative regime to the nonperturbative regime, none of the observables we have considered shows an anomalous behavior, so that we conclude that there is no sharp phase transition between them.

For all the values of $\alpha$ we have considered, at some $\theta_c(\alpha)$ the system undergoes a second order phase transition to a predominantly Néel ordered state aligned in the $x$ direction.

The first excited states in these phases are kinks. The gap to them vanishes as $\alpha$ approaches zero. At $\alpha = 0$, indeed, the 1D geometry of the system is completely lost and the Néel state melts into an exponentially degenerate groundstate subspace made of all possible arrangements of $N/2|+\rangle$ states and $N/2|-\rangle$, where $|+\rangle$ and $|-\rangle$ are the eigenvectors of $\sigma_x$.

The value of $\theta_c(\alpha)$ is always larger than the one of the ITF transition at $\pi/4$. Intuitively, the slower the two body interaction decays, (smaller $\alpha$) the more the $\sigma_x$ part of the Hamiltonian becomes frustrated. As a consequence, for small values of the transverse field (large values of $\theta$) the $z$ polarized state has a lower energy. Indeed, in the $(\theta, \alpha)$, $\theta_c$ increases with decreasing $\alpha$.

The phase diagram is in Fig. 1, where we plot $S_{L/2}$ of Eq. (2) as a function of both $\alpha$ and $\theta$. For fixed $L$ and $\alpha$, $S_{L/2}$ has a maximum at $\theta^*$. The extrapolation of $\theta^*$ as a function of $L$ unveils the location of the critical point $\theta_c^{\infty}(\alpha)$. These points are superimposed in black in Fig. 1.

In the $z$ polarized phase, we observe two striking phenomena. On one side, even if the phase is gapped, we observe polynomially decaying correlation functions. Namely, $\langle \sigma_x^{L/2} \sigma_x^{L/2+r} \rangle_c \propto r^{-\alpha}$ while $\langle \sigma_z^{L/2} \sigma_z^{L/2+r} \rangle_c \propto r^{-2\alpha}$ [36] (similar results were also obtained in Refs. [17,18]). On the other side we observe violations to the area law for the entanglement entropy, as the entropy of half chain increases with the size of the chains. There are two regimes for the violations depending on the value of $\alpha$. For $\alpha \leq 1$ they are logarithmic so that, by using a tempting analogy with the case of critical systems, [37–39] we can define an "effective central charge" as

$$S_{L/2} \propto \frac{c}{6} \log L. \quad (5)$$

The value we determine for $c/6$ are reported in the upper panel of Fig. 2. The small dispersion of the curves obtained

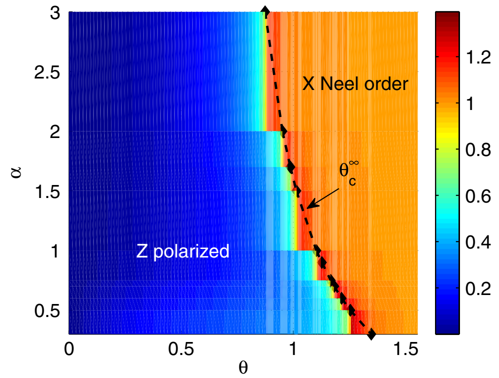


FIG. 1 (color online). Phase diagram of the LITF from the entanglement entropy. The half chain entanglement entropy provides information about the phase diagram of 1 as a function of $\theta$ and $\alpha$. The background colors represent the value $S_{L/2}$ for a system of size $L = 100$. The maximum of it, signals the vicinity of a phase transition. Extrapolating its position as a function of $L$, for $L = 20, \ldots, 100$, we locate the position of the transition in the thermodynamic limit $\theta_c^\infty$. The results are superimposed as solid black dots connected by a dashed line.

from different system sizes $L$ around a single curve is a confirmation of the correctness of the scaling form 5. Interestingly, this effective central charge, in the nonperturbative regime, varies very slowly with $\theta$. For $\alpha > 1$ we still observe a steady growth of the entanglement with the size of the blocks but its behavior is sublogarithmic. Our data for $\alpha = 3$ are not conclusive, they suggest the presence of sublogarithmic corrections but we cannot exclude that the entropy would eventually saturate for larger systems. We leave this as an open issue.

The ES of Eq. (3) can be used to distinguish between the perturbative and the "nonperturbative" regime in the $z$ polarized phase. In the perturbative regime, (for $\theta \simeq 0$), the ES shows well defined scale separation, proportional to different powers of the small parameter $\theta$. The elements of the spectrum are dominated by their leading order in $\theta$. In the ITF there is a single element at each order in PT, whereas in the LITF ES instead, multiple eigenvalues appear at the same order in PT. They can be identified as parallel straight lines by plotting the ES in a log-log plot as a function of $\theta$. The slopes of them indicate to which order in PT the eigenvalue belongs to, as shown in Fig. 3 right panel for $\alpha = 2$. There we appreciate both the proliferation of eigenvalues and the wide range of validity of PT. We also see that the ES is dominated by eigenvalues appearing at most at order $\theta^4$ in PT.

In the same range of $\theta$, the ES for $\alpha = 0.3$ looks very different. In Fig. 3 right panel, we do not see neither a clear separations of scales, nor a well-defined power-law behavior of the eigenvalues with respect to $\theta$ both footprints of the "nonperturbative" regime [40]. The eigenvalues tend to cluster in bands (and the respective gaps) that are robust to changes in size (at least for the range of sizes we have access to). An unsolved issue is whether they would survive to the thermodynamic limit.

In both perturbative and nonperturbative regime the logarithmic violations to the area law coexist with a gapped ES. This gap is likely to survive in the thermodynamic limit, so that these corrections are different from those of a quantum critical point, where the ES gap closes with the system size [41].

*The phase transition.*—For the anti-ferromagnetic interaction we find a phase transition for every value of $\alpha > 0$. This has to be compared with the ferromagnetic case where there is a lower critical dimension $\alpha = 1/2$ [22]. A mean

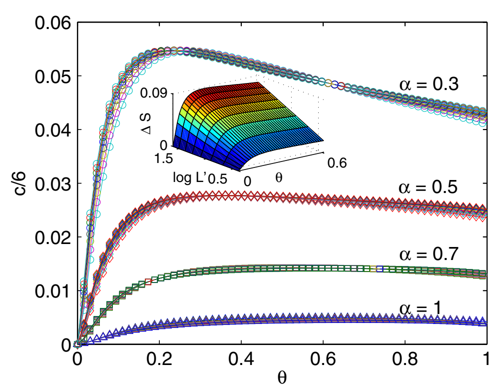


FIG. 2 (color online). Violations to the area law. The entanglement entropy for a bipartition increases monotonically with the system size in the whole $z$ polarized phase. For $\alpha \leq 1$ the entropy scales logarithmically with the size of the system (small inset). The prefactor in Eq. (5) is extracted by plotting $c/6 = \frac{\Delta S}{\log L'}$, with $\Delta S = S_{L/2} - S_{L_0/2}$ and $L' = L/L_0$, $L_0 = 20$, $L = 30, \ldots, 100$ (we use as a reference size $L_0$ in order to eliminate the constant terms in the scaling).

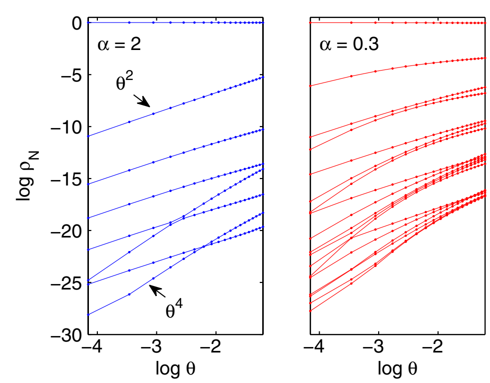


FIG. 3 (color online). Structure of the ES defined in Eq. (3) as a function of $\log(\theta)$, for $0 \leq \theta \leq 0.36$, deep in the $z$ polarized phase. Left panel $\alpha = 2$, the spectrum presents well separated scales reproducible by the lowest order in PT. Several eigenvalues belong to the same order in PT (that can be distinguished as parallel lines) contrary to what happens in the ITF. Right panel, the "nonperturbative" regime for $\alpha = 0.3$, there is no clear scale separation, and no clear power law dependence of the eigenvalues with $\theta$. The eigenvalues tend to cluster to form bands separated by gaps.

field analysis around the ITF critical point [3,42,43] suggests that the LR interactions are relevant for $\alpha < 2 + 2\Delta_x^{\rm SR}$ driving the system to a different critical point than the SR case, with $\Delta_x^{\rm SR} = 1/8$ being the scaling dimension of the $\sigma_x$ operator for the SR ITF. For $\alpha = 2.25$ the LR is marginal, while for $\alpha > 2.25$ it becomes irrelevant and one should observe the standard SR ITF criticality.

We check the above statements performing a finite-size scaling analysis of the correlation functions 4, $\langle \sigma_x^{L/2}, \sigma_{xc}^{L/2+L/5}\rangle_c \propto L^{-2\Delta_x^{\rm LR}}$. The exponents $\Delta_x^{\rm LR}(\alpha)$ are presented in the upper panel of Fig. 4, $2\Delta_x^{\rm LR}(\alpha)$ is different from $2\Delta^{\rm SR}$ for all values of $\alpha < 2$ while between 2 and 3 it becomes very close to expected SR value $1/4$.

By studying the scaling of $S_{L/2}$ in Eq. (5), we can extract the value of the central charge of the corresponding CFT that, for the ITF, is $c = 1/2$. In the whole range of $\alpha$ considered, the coefficient we obtain is systematically bigger than $1/2$. The reason for that is not clear but probably resides in a mixture of effects: (i) the effects of boundaries are enhanced by the LR interaction; (ii) the system sizes we can address are still too small to get rid of the irrelevant contributions to the leading scaling [44] (indeed our data agree with a pure logarithmic scaling only for the biggest lattices $L = 70, \ldots, 100$); (iii) the LR could induce marginal operators whose corrections to the scaling are difficult to control [44]. The corresponding plot is presented in the lower panel of Fig. 4.

Finally, we have checked the leading power-law scaling of the $\sigma_z$ correlation. In the ITF its leading scaling is dictated by the thermal exponent $\Delta_z^{\rm SR} = 1$. The results for the LITF are presented in the central panel of Fig. 4. In the SR regime, for $\alpha > 2.25$, the exponent we extract from the fit gives an estimate of the thermal exponent off by around 10% clear symptom of contamination with sub-leading corrections.

*Conclusions and outlook.*—In this Letter we have considered the effects of a LR antiferromagnetic interaction on the phase diagram of the ITF, in order to both provide a guide to future trapped ions experiments and study the increase of complexity induced by the LR interactions. The resulting phase diagram shows that the frustration favors the $z$ polarized phase over the $x$ aligned Néel phase. For all values of $\alpha > 0$ considered we have located the phase transition. There we have confirmed that the LR interaction is relevant for $\alpha \leq 2.25$, inducing critical exponents different from the ones of the ITF. We have determined them for the $\sigma_x\sigma_x$ and $\sigma_z\sigma_z$ correlations (the equivalent of the magnetic and thermal exponents in the SR case). They vary continuously as a function of $\alpha$ in the range $0 < \alpha < 2.25$. The scaling of the entanglement entropy in the SR regime is used to provide an estimate for the central charge $c$ of the underlying CFT, that turns out to be systematically larger than the expected value $1/2$. We miss a complete understanding of this result (that however could be a manifestation of the fact that the systems we can address are still too small to see the expected asymptotic scaling) and further studies should be devoted to clarify it.

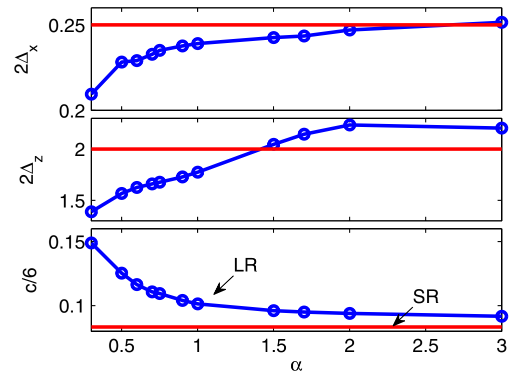


FIG. 4 (color online). Long-range universality class. From a mean field analysis for $\alpha > 2.25$ the LR become irrelevant. Upper panel, $2\Delta_x^{\rm LR}$ as a function of $\alpha$. As expected close to $\alpha = 2$ the exponent tends to its SR value $1/4$. Middle panel, $2\Delta_z^{\rm LR}$ does not coincide with the expected thermal exponent 2 due to sub-leading corrections. Lower panel. The central charge extracted from $S_{L/2}$, unexpectedly, is systematically larger than $1/2$.

The complexity of the ground state induced by the LR part of the Hamiltonian is not significantly higher than the one of their SR equivalent. There are violations to the area law for the entanglement entropy, whose strength depends on $\alpha$. The strongest violations are found for $\alpha \leq 2$ and are logarithmic in the system size. These violations appear only in the SR dominated $z$ polarized, gapped phase. They seem to be always accompanied by a finite entanglement gap and in some cases by the presence of bands in the ES. Further studies should be devoted to check the persistence of the corrections for dipolar interactions in the thermodynamic limit. In the $x$ aligned Néel phase, dominated by the LR part of the Hamiltonian, there are no violations to the area law.

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