Avalanche Collapse of Interdependent Networks

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We reveal the nature of the avalanche collapse of the giant viable component in multiplex networks under perturbations such as random damage. Specifically, we identify latent critical clusters associated with the avalanches of random damage. Divergence of their mean size signals the approach to the hybrid phase transition from one side, while there are no critical precursors on the other side. We find that this discontinuous transition occurs in scale-free multiplex networks whenever the mean degree of at least one of the interdependent networks does not diverge.

Many complex systems, both natural [1] and man made [2,3], can be represented as multiplex or interdependent networks. Multiple dependencies make a system more fragile: damage to one element can lead to avalanches of failures throughout the system [4,5]. Recent theoretical investigation of two [6] or more [7] networks in which vertices in each network mutually depend on vertices in other networks has shown that indeed small initial failures can cascade back and forth through the networks, leading to a discontinuous collapse of the whole system. Damage in one network propagates along edges and leads to damage in the other network. This is an individual stage of a cascade in back-and-forth damage propagation. Son *et al.* [8] showed that this approach can be simplified and is equivalent to considering damage propagation in multiplex networks. They proposed a simple mapping between the model used in Ref. [6] in which a vertex in one network has a mutual dependence on exactly one vertex in the other network, and a multiplex network with one kind of vertex but two kinds of edges. The mapping is achieved by simply merging the mutually dependent vertices from the two networks.

In this Letter we describe the nature of such discontinuous phase transitions. We consider a set of vertices connected by m different types of edges (dependencies). The connections are essential to the function of each site, so that a vertex is only viable if it maintains connections of every type to other viable vertices. A *viable cluster* is defined as follows: For every kind of edge, and for any two vertices i and j within a viable cluster, there must be a path from i to j following only edges of that kind. A graph containing two finite viable clusters is illustrated in Fig. 1. We wish to find when there is a giant cluster of viable vertices. Note that any giant viable cluster is a subgraph of the giant connected component of each of the m networks.

Various parameters can be used to control the critical behavior of this system: the mean degrees of the networks, amount of random damage, and so on. Small perturbations to the system can propagate, leading to avalanches of

further damage. In uncorrelated, random networks we find a discontinuous hybrid transition in the collapse of the giant viable cluster, similar to that seen in the k-core or bootstrap percolation [9,10]. In Ref. [6], the propagation of damage caused by removal of a finite fraction of vertices in one of the interdependent networks was studied. In contrast, we study the avalanches of damage triggered by the removal of randomly chosen single vertices. These avalanches increase in size approaching the critical point, signaling the impending collapse of the giant viable cluster. At the critical point the mean avalanche size diverges. Below the transition, on the other hand, there is no precursor for the appearance of the giant viable cluster. The transition is thus asymmetric. It is hybrid in nature, having a discontinuity like a first-order transition, but exhibiting critical behavior, only above the transition, like a second-order transition. A complete understanding of the transition cannot therefore be had without first understanding this critical behavior. We have discovered critical clusters which collapse in avalanches of diverging size as the transition is approached. These critical clusters are thus responsible for both the critical scaling and the discontinuity observed in the size of the giant viable cluster. As we shall see, the critical clusters have a novel character as, unlike the corona clusters of the k-core; for example [9], avalanches propagate in a directed way through critical clusters. The critical clusters may have important practical applications, helping to identify vulnerabilities to targeted attack, as well as informing efforts to guard against such attack. Surprisingly, when the degree distributions are
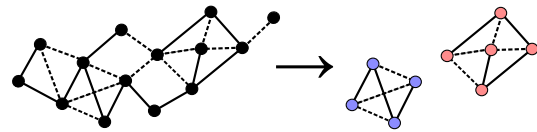

FIG. 1 (color online).   A small network with two kinds of edges (left). Applying the algorithm described in the text, nonviable vertices are removed leaving two viable clusters (right).

asymptotically power laws $P(q) \propto q^{-\gamma}$ the critical point p_c (taking the undamaged fraction of vertices p as the control parameter) remains at a finite value even when the exponents γ of the degree distributions are below 3, remaining finite until both exponents reach 2, in agreement with an argument given in Ref. [6]. This is in stark contrast to ordinary percolation in complex networks, in which the threshold falls to zero as soon as γ reaches 3 [11,12]. We show, further, that the nature of the transition doesn't change. Although the height of the discontinuity becomes extremely small near $\gamma = 2$, it remains finite near this limit (see Fig. 2).

Algorithm.—We consider a multiplex network, with vertices $i = 1, 2, \ldots, N$ connected by m kinds of edges labeled $s = a, b, \ldots$. The joint degree distribution is $P(q_a, q_b, \ldots)$. Viable clusters in any multiplex network may be identified by the following algorithm. (i) Choose a test vertex i at random from the network. (ii) For each kind of edge s, compile a list of vertices that can be reached from i by following only edges of type s. (iii) The intersection of these m lists forms a new candidate set for the viable cluster containing i. (iv) Repeat steps (ii) and (iii) but traversing only the current candidate set. When the candidate set no longer changes, it is either a viable cluster, or contains only vertex i. (v) To find further viable clusters, remove the viable cluster of i from the network (cutting any edges) and repeat steps (i)–(iv) on the remaining network beginning from a new test vertex.

Repeated application of this procedure will identify every viable cluster in the network. The application of this procedure to a finite graph is illustrated in Fig. 1.

Basic equations.—Let us consider the case of sparse uncorrelated networks, which are locally treelike in the infinite size limit $N \rightarrow \infty$. In such a network there are no finite viable clusters. In order to find the giant viable cluster, we define X_s, with $s \in \{a, b, \ldots\}$, to be the probability that, on following an arbitrarily chosen edge of
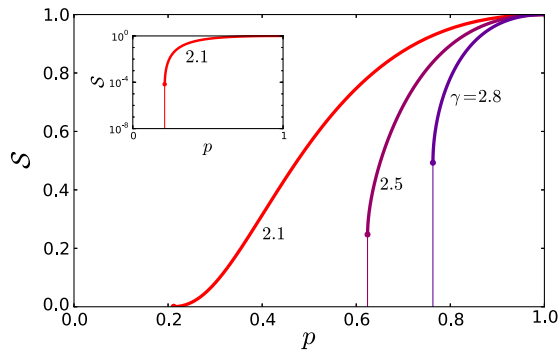
type s, we encounter the root of an infinite subtree formed solely from type s edges, whose vertices are also each connected to at least one infinite subtree of every other type. We call this a type s infinite subtree. The vector $\{X_a, X_b, \ldots\}$ plays the role of the order parameter. A vertex is then in the giant viable cluster if it has at least one edge of every type s leading to an infinite type s subtree (probability X_s), as shown in Fig. 3(a). Using the locally treelike property of the networks, we can write self-consistency equations for the probabilities X_s:

$$X_s = \Psi_s(X_a, X_b, \ldots)$$
$$\equiv \sum_{q_a, q_b, \ldots} \frac{q_s}{\langle q_s \rangle} P(q_a, q_b, \ldots)[1 - (1 - X_s)^{q_s - 1}]$$
$$\times \prod_{l \neq s}[1 - (1 - X_l)^{q_l}] \qquad (1)$$

for each $s \in \{a, b, \ldots\}$. This is illustrated in Fig. 4. The term $(q_s/\langle q_s \rangle)P(q_a, q_b, \ldots)$ gives the probability that on following an arbitrary edge of type s, we find a vertex with degrees $q_a, q_b, \ldots$, while $[1 - (1 - X_a)^{q_a}]$ is the probability that this vertex has at least one edge of type $a \neq s$ leading to the root of an infinite subtree of type a edges (i.e., probability X_a). This becomes $[1 - (1 - X_s)^{q_s - 1}]$ when $a = s$. The argument leading to Eq. (1) is similar to that used in Ref. [8]. Solving these equations enables us to calculate all the quantities of interest. In particular, the relative size of the giant viable cluster is

$$\mathcal{S} = \sum_{q_a, q_b, \ldots} P(q_a, q_b, \ldots) \prod_{s=a, b, \ldots} [1 - (1 - X_s)^{q_s}], \qquad (2)$$

which is illustrated in Fig. 3(a).

A hybrid transition appears at the point where $\Psi_s(X_a, X_b, \ldots)$ first meets X_s at a nonzero value, for all s. This occurs when

$$\det[\mathbf{J} - \mathbf{I}] = 0, \qquad (3)$$

where $\mathbf{I}$ is the unit matrix and $\mathbf{J}$ is the Jacobian matrix $J_{ab} = \partial \Psi_b / \partial X_a$. Expanding Ψ_s about the critical point, at which Eqs. (1) and (3) are both satisfied, we find the scaling of X_s and hence $\mathcal{S}$, the size of the giant viable cluster. For example, random damage can be considered by introducing a parameter p, the fraction of vertices remaining undamaged. This is incorporated by multiplying the right-hand sides of Eqs. (1) and (2) by a factor p. Then

FIG. 2 (color online). Size of the giant viable cluster $\mathcal{S}$ as a function of the fraction p of vertices remaining undamaged for two symmetric power-law distributed networks with, from right to left, $\gamma = 2.8$, 2.5, and 2.1. The height of the jump becomes very small as γ approaches 2, but is not zero, as seen in the inset, which is $\mathcal{S}$ vs p on a logarithmic vertical scale for $\gamma = 2.1$.
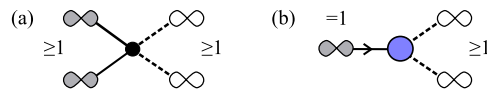

FIG. 3 (color online). Viable and critical viable vertices for two interdependent networks. (a) A vertex is in the giant viable cluster if it has connections to giant viable subtrees (represented by infinity symbols) of both kinds. (b) A critical viable vertex of type a has exactly 1 connection to a giant subtree of type a.

FIG. 4.   Diagrammatic representation of Eq. (1) in a system of two interdependent networks a and b. The probability X_a, represented by a shaded infinity symbol, can be written recursively as a sum of second-neighbor probabilities. Open infinity symbols represent the equivalent probability X_b for network b, which obeys a similar recursive equation.

$$\mathcal{S} - \mathcal{S}_c \propto X_s - X_s^{(c)} \propto (p - p_c)^{1/2}. \tag{4}$$

A similar result is found for other control parameters.

Avalanches.—To examine the hybrid transition we focus on the case of two types of edges. Consider a viable vertex that has exactly one edge of type a leading to a type a infinite subtree, and at least one edge of type b leading type b infinite subtrees. We call this a critical vertex of type a. It is illustrated in Fig. 3(b). Critical vertices of type a will drop out of the viable cluster if they lose their single link to a type a infinite subtree. We mark these special edges with an arrow leading to the critical vertex. An avalanche can only transmit in the direction of the arrows. A vertex may have outgoing edges of this kind, so that removal of this vertex from the giant viable cluster also requires the removal of the critical vertices which depend on it. For example, in Fig. 5, removal of the vertex labeled 1 removes the essential edge of the critical vertex 2 which thus becomes nonviable. Removed critical viable vertices may in turn have outgoing critical edges, so that the removal of a single vertex can result in an avalanche of removals of critical vertices from the giant viable cluster. In Fig. 5, removal of 2 causes the removal of further critical vertices 3 and 4, and the removal of 4 then requires the removal of 5. Thus critical vertices form critical clusters. At the head of each critical cluster is a "keystone vertex" (e.g., vertex 1 in the figure) whose removal would result in the removal of the entire cluster. Graphically, upon removal of a vertex, we remove all vertices found by following the arrowed edges. As we approach the critical point (from above),
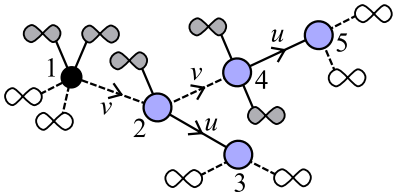

FIG. 5 (color online).   A critical cluster. Removal of any of the shown viable vertices will result in the removal of all downstream critical viable vertices. Removal of the vertex labeled 1 will result in all of the shown vertices being removed (becoming nonviable), while removal of vertex 4 results only in vertex 5 also being removed.

diverging avalanches cause a discontinuity in the size of the giant viable cluster, which collapses to zero.

There are three possibilities when following an arbitrarily chosen edge of a given type: (i) with probability X_s we encounter a type s infinite subtree; (ii) with probability R_s we encounter a vertex which has a connection to an infinite subtree of the opposite type, but none of the same type. Such a vertex is part of the giant viable cluster if the parent vertex was; or (iii) with probability $1 - X_s - R_s$, we encounter a vertex which has no connections to infinite subtrees of either kind. The probability R_a obeys

$$R_a = \sum_{q_a} \sum_{q_b} \frac{q_a}{\langle q_a \rangle} P(q_a, q_b)(1 - X_a)^{q_a - 1}[1 - (1 - X_b)^{q_b}], \tag{5}$$

and similarly for R_b. We use generating functions to examine the sizes of critical clusters. We first define the function $F_a(x, y)$ as

$$F_a(x, y) = \sum_{q_a} \sum_{q_b} \frac{q_a}{\langle q_a \rangle} P(q_a, q_b) x^{q_a - 1} \sum_{r=1}^{q_b} \binom{q_b}{r} X_b^r y^{q_b - r}, \tag{6}$$

and similarly for $F_b(x, y)$, by exchanging all subscripts a and b. The generating function for the size of an avalanche triggered by removing an arbitrary type a edge which does not lead to an infinite type a subtree can be defined in terms of these functions by

$$H_a(u, v) = 1 - X_a - R_a + u F_a[H_a(u, v), H_b(u, v)] \tag{7}$$

and similarly for $H_b(u, v)$. This recursive equation can be understood by noting that $H_a(0, v) = 1 - X_a - R_a$ is the probability that an arbitrarily chosen edge leads to a vertex outside the viable cluster. Here u and v are auxiliary variables. Following through a critical cluster, a factor u appears for each arrowed edge of type a, and v for each arrowed edge of type b. For example, the critical cluster illustrated in Fig. 5 contributes a factor $u^2 v^2$. The mean number of critical vertices reached upon following an edge of type a, i.e., the mean size of the resulting avalanche if this edge is removed, is given by $\partial_u H_a(1, 1) + \partial_v H_a(1, 1)$. Unbounded avalanches emerge at the point where $\partial_u H_a(1, 1)$ [or $\partial_v H_b(1, 1)$] diverges. Taking derivatives of Eq. (7), and using that $H_a(1, 1) = 1 - X_a$ and $F_a(1 - X_a, 1 - X_b) = R_a$, and that, from Eqs. (1) and (6), $\partial_x F_a(1 - X_a, 1 - X_b) = \partial_a \Psi_a(X_a, X_b)$ and $\partial_y F_a(1 - X_a, 1 - X_b) = (\langle q_a \rangle / \langle q_b \rangle) \partial_a \Psi_b(X_a, X_b)$, gives

$$\partial_u H_a(1, 1) = \frac{R_a[1 - \partial_b \Psi_b(X_a, X_b)]}{\det[\mathbf{J} - \mathbf{I}]}. \tag{8}$$

From Eq. (3) we see immediately that this diverges at the critical point, meaning that the mean size of avalanches triggered by random removal of vertices diverges exactly at the point of the hybrid transition. The mean size of the avalanches can be related to the susceptibility, of the giant

viable cluster to random damage, similar to the susceptibility for ordinary percolation [13].

Because of the similarity of Eq. (4) to the k-core version [14], we can expect that the size distribution of avalanches triggered by randomly removed vertices obeys power law $p(s) \propto s^{-\sigma}$ with $\sigma = 3/2$.

Scale-free networks.—In ordinary percolation, and even the k-core problem, networks with degree distributions that are asymptotically power laws $P(q) \sim q^{-\gamma}$ may exhibit qualitatively different transitions, especially when $\gamma < 3$. To investigate such effects in the giant viable cluster, we consider two uncorrelated scale-free networks, so $P(q_a, q_b) = P_a(q_a)P_b(q_b)$, having power-law degree distributions with fixed minimum degree $q_0 = 1$ (then $\langle q \rangle \approx (\gamma - 1)q_0/(\gamma - 2)$), so that $P_s(q_s) = \zeta(\gamma_s)q^{-\gamma_s}$, where s takes the values a or b. As a control parameter we apply random damage to the system as a whole so that vertices survive with probability p. First consider the case that $\gamma_a < 3$ or $\gamma_b < 3$ (or both).

The giant viable cluster is necessarily a subgraph of the overlap between the giant components of each graph. We know from ordinary percolation that for $\gamma > 3$, the giant component appears at a finite value of p [15]. It follows that the giant viable cluster, also, cannot appear from $p = 0$; there must be a finite threshold p_c, (with a hybrid transition) This is true even if one of the networks has $\gamma_s < 3$.

The more interesting case is when γ_a, $\gamma_b < 3$, when the percolation threshold is zero for each network when considered separately. Let us write $\gamma_a = 2 + \delta_a$ and $\gamma_b = 2 + \delta_b$, and examine the behavior for small δ_a and δ_b. We proceed by assuming that in this situation, for p near p_c, Eqs. (1) have a solution with small X_a, $X_b \ll 1$. Writing only leading orders,

$$\Psi_a(X_a, X_b) = p\frac{\pi^2}{6\delta_b}X_a^{\delta_a}(X_b - X_b^{1+\delta_b}) \qquad (9)$$

and similarly for $\Psi_b(X_a, X_b)$. The location of the critical point is found from Eq. (3) which becomes

$$\delta_a + \delta_b = p\frac{\pi^2}{6}X_a^{\delta_a}X_b^{\delta_b}\left(\frac{X_a}{X_b} + \frac{X_b}{X_a}\right). \qquad (10)$$

Substituting Eq. (9) into Eq. (1) and solving with Eq. (10) we find X_s and $\mathcal{S}$ at p_c. We find in general that the hybrid transition persists for δ_a, $\delta_b \neq 0$, though the height of the discontinuity at the hybrid transition becomes extremely small for δ small. In experiments or simulations, this could be misinterpreted as evidence of a continuous phase transition. We describe two representative cases. First, where $\delta_a \ll \delta_b$, that is, $\gamma_a \to 2$ while $\gamma_b > 2$. We find that $p_c \approx 1.19\delta_b$, and the size of the giant viable cluster at p_c is $\mathcal{S}_c = Ae^{-B/\delta_b}$ with $A \approx 3.36$ and $B \approx 2.89$. We see that a hybrid transition occurs, albeit with an extremely small discontinuity, at a nonzero threshold p_c as long as at least one of δ_a and δ_b is not equal to zero. To examine the

case that both tend to zero, we consider the symmetric case $\delta_a = \delta_b \equiv \delta$. Then $X_a = X_b \equiv X$, and the discontinuity is found by requiring $\Psi'(X) = 1$ [from Eq. (3)]. We find that $X_c = (1/2)^{1/\delta}$, $p_c = 24\delta/\pi^2$, and, $\mathcal{S}_c = 4^{1-1/\delta}$. The critical point, p_c, tends to 0 as $\delta \to 0$, and $\mathcal{S}_c$ becomes very small even for nonzero δ, but vanishes completely as $\delta \to 0$; see Fig. 2. Expanding $\Psi(X)$ about X_c we find again square-root scaling, $X/X_c - 1 = A(p/p_c - 1)^{1/2}$ with $A = 12/\pi^2\delta p_c$, which holds while $p - p_c \ll \delta^3$.

Summary.—We have given an algorithm for identifying the viable clusters in any multiplex network. Under increasing damage, the giant viable cluster collapses in a discontinuous hybrid transition, in contrast to the smooth continuous transition found in simplex networks. We have shown that this transition is signaled by avalanches triggered by removing vertices at random. The mean size of the avalanches diverges as the collapse approaches. To understand this critical behavior, which occurs only above the transition, we successfully identified clusters of critical vertices. These clusters determine the structure and statistics of avalanches of damage. Avalanches sweep through the critical clusters in a directed fashion, and it is the diverging size of these clusters which accounts for the criticality. This directed nature stands in contrast to, for example, the corona clusters found in the k-core problem [14]. Each critical cluster depends upon a keystone vertex whose removal completely destroys the critical cluster. These keystone vertices are good candidates for targeted attack or immunization against such attacks.

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