Topology, Delocalization via Average Symmetry and the Symplectic Anderson Transition

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A field theory of the Anderson transition in two-dimensional disordered systems with spin-orbit interactions and time-reversal symmetry is developed, in which the proliferation of vortexlike topological defects is essential for localization. The sign of vortex fugacity determines the Z_2 topological class of the localized phase. There are two distinct fixed points with the same critical exponents, corresponding to transitions from a metal to an insulator and a topological insulator, respectively. The critical conductivity and correlation length exponent of these transitions are computed in an $N = 1 - \epsilon$ expansion in the number of replicas, where for small ϵ the critical points are perturbatively connected to the Kosterlitz-Thouless critical point. Delocalized states, which arise at the surface of weak topological insulators and topological crystalline insulators, occur because vortex proliferation is forbidden due to the presence of symmetries that are violated by disorder, but are restored by disorder averaging.

Topology can have a profound impact on Anderson localization in disordered electronic systems. This was first understood in the integer quantum Hall effect [1,2], where the two-dimensional (2D) bulk states at the plateau transition are extended, even in the presence of strong disorder. Subsequently it was recognized that topological insulators (TI) exhibit boundary states that similarly remain extended in the presence of time-reversal (TR) invariant disorder [3–5]. In the field theory of localization, this delocalization is associated with the presence of topological terms in the nonlinear σ model (NLσM) [6–10].

A shortcoming of the conventional scaling theory of localization [11–14] is that it involves only a single parameter, the conductivity. It cannot distinguish the trivial insulator from the TI, and it does not explain the metallic phase that generically occurs between them [15–17]. A related difficulty is revealed by recent studies of surface states of 3D weak TIs (WTI) [18–20] and of topological crystalline insulators (TCI) [21,22]. General arguments, as well as numerics, suggest that these surfaces remain delocalized even with strong disorder, due to symmetries that are violated by disorder, but remain unbroken on average. This led to the suggestion that there should be a second symmetry breaking parameter in the scaling theory [19]. This poses the question of how average symmetries fit into the field theory of localization, and what the role of the second parameter is.

In this Letter we answer those questions by examining the crucial and largely unexplored role played by topological defects in the NLσM in the 2D symplectic class [23]. We show that localization is driven by the proliferation of pointlike Z_2 vortices, and that the sign of the vortex

fugacity distinguishes a TI from a trivial insulator. We find that average symmetries can place the system on a line where the vortex fugacity vanishes, dictating delocalization. This analysis also provides new insight into the 2D symplectic metal-insulator transition. We find two distinct but equivalent fixed points describing transitions to insulator and TI states. By treating the number of replicas, N as a continuous variable, we show that for $N = 1 - \epsilon$ the Anderson transition fixed points are perturbatively connected to the Kosterlitz-Thouless (KT) transition fixed point [24] for $\epsilon \to 0$. This allows us to compute the critical conductivity and correlation length exponent perturbatively in an ϵ expansion.

Before describing the symplectic class, we briefly discuss a simpler version of delocalization via average symmetry in the unitary class. The surface of a 3D strong TI (STI) is delocalized [9,25], but TR violating perturbations lead to localization. The localized state is in a sense "half" of a quantum Hall state and has $\sigma_{xy} = \pm e^2/2h$. Importantly, the time reverse of this state, with $\sigma_{xy} = \mp e^2/2h$, is topologically distinct. If impurities have random local moments so TR symmetry is unbroken on average then the system is precisely at the transition between the two localized states. This can be modeled by an ensemble in which each member violates TR, but the whole ensemble is TR invariant. This is described by a NLσM in the unitary class, which in 2D allows a topological term [6] characterized by an angle θ related to the Hall conductivity. Since TR, applied to the ensemble, takes θ to $-\theta$, average TR symmetry constrains θ to be 0 or π. The surface of a STI corresponds to $\theta = \pi$, so the surface is precisely at the critical point of the quantum Hall plateau

transition [26]. If the average symmetry is broken by an applied magnetic field, then the system flows to a localized phase with $\sigma_{xy} = \pm e^2/2h$.

WTI and TCI surfaces also have discrete average symmetries. For a layered WTI it is translation by one layer. For the TCI studied in Ref. [22], it is a mirror symmetry. Breaking the symmetry gaps the surface, leading to localization. However, applying the symmetry to the gapped state leads to a topologically distinct localized state, so that there exists a 1D helical edge mode at the interface between the two localized states. If the symmetry is respected on average, then the system is at the boundary between the two localized states. It is clear that even for strong disorder, the system cannot be localized at this point because a change in topological class can only occur when extended states are present at the Fermi energy.

To develop a field theory for this delocalization, we use the fermionic replica theory introduced by Efetov et al. [13]. Our analysis closely parallels that of Ryu et al. [9]. We consider a system with average Hamiltonian $\mathcal{H}_0$ and Gaussian correlated TR invariant disorder. Using the replica trick, the disorder averaged product of retarded and advanced Green's functions can be generated from the partition function $Z = \int D[\bar{\psi}, \psi] e^{-S}$, with

$$S = \int d^2 r \Big[\bar{\psi}_a [(\mathcal{H}_0 - E)\delta_{ab} + i\eta \Lambda_{ab}]\psi_b$$
$$- \frac{g}{2}(\bar{\psi}_a \psi_b)(\bar{\psi}_b \psi_a) \Big]. \tag{1}$$

Here $a = 1, \ldots, 2N$ is an index for N retarded and N advanced replicas, and $\Lambda = 1_N \oplus (-1)_N$, where 1_N is an $N \times N$ identity matrix. g is a coupling constant that characterizes the disorder strength, and ψ_a is a Grassmann field that includes (suppressed) spin, position, and possibly orbital indices. $\bar{\psi}_a \equiv \psi_a^T i\sigma^y$, where σ acts on the spin indices. TR requires $\sigma^y \mathcal{H}_0 \sigma^y = \mathcal{H}_0^*$, so $i\sigma^y(H_0 - E)$ is a skew symmetric matrix. For $\eta = 0$, (1) is invariant under $O(2N)$ rotations among the replicas, which is broken down to $O(N) \times O(N)$ by η.

A theory of the Nambu Goldstone modes associated with this symmetry breaking is formulated by Hubbard Stratonovich decoupling the four fermion interaction and performing a saddle point expansion about the broken symmetry state. After freezing the massive modes, the saddle point is characterized by a $2N \times 2N$ matrix field $Q = O^T \Lambda O$, with $O \in O(2N)$. Distinct values of Q belong to the coset $G/H = O(2N)/O(N) \times O(N)$ and satisfy $Q = Q^T$, $Q^2 = 1$. A theory for the long wavelength fluctuations in Q_{ab} is obtained by integrating ψ_a in the background of a spatially varying Q_{ab}. This gives $Z_{eff} = \int D[Q]e^{-S_{eff}[Q]}$ with

$$e^{-S_{eff}[Q]} = \int D[\bar{\psi}, \psi] e^{-\int d^2 r[\bar{\psi}_a[(\mathcal{H}_0 - E)\delta_{ab} + i\Delta Q_{ab}]\psi_b]}. \tag{2}$$

Here Δ is a parameter characterizing the bare scattering time that is determined self-consistently at the saddle point. Expanding in gradients gives the NLσM,

$$S_{eff}^0[Q] = \frac{1}{32\pi t} \int d^2 r \, \text{Tr}[(\nabla Q)^2], \tag{3}$$

where the coupling constant t characterizes the disorder strength and is related at lowest order to the resistivity, $\sigma = (2\pi t)^{-1}e^2/h$. The renormalization of t at long wavelengths is described by the perturbative renormalization group (RG) equation [13,14,27–29]

$$dt/d\ell = \beta(t), \qquad \beta(t) = 2(N - 1)t^2 + \cdots. \tag{4}$$

In the replica limit, $N \to 0$, the weak coupling fixed point $t = 0$ is stable, indicating the stability of the symplectic metal phase, characterized by weak antilocalization.

Equation (3) is not the whole story because topologically nontrivial configurations of Q can have important nonperturbative effects. There are two types of topological configurations associated with the nontrivial homotopy groups $\pi_1(G/H) = \pi_2(G/H) = Z_2$ [10]. $\pi_2(G/H)$ allows a topological term that prevents localization on the surface of 3D TI [8,9]. That term is absent in purely 2D systems as well as WTI or TCI surfaces. For our problem, the crucial topological objects are pointlike defects similar to vortices that are allowed by the nontrivial $\pi_1(G/H)$. These defects are necessary for localization, and their contribution to Z_{eff} encodes the distinction between a trivial insulator and TI.

The role of vortices can be understood by considering an inhomogeneous 2D system in which a TI in region S with boundary C is surrounded by a trivial insulator [Fig. 1(a)]. Imagine integrating out ψ_a in (2) in the presence of a vortex configuration $Q_{ab}(\mathbf{r})$. Since the interior of S has a finite gap, the dominant contribution comes from the 1D helical edge states at the boundary C. On C, $Q_{ab}(\mathbf{r} \in C)$ is a nonsingular and noncontractible configuration corresponding to the nontrivial element of $\pi_1(G/H)$. Repeating the analysis of Ryu et al. [9] for 1D helical states, we find a topological term in the 1D NLσM [10],

$$e^{-S_{eff}[Q]} \propto (-1)^{n(C)}, \tag{5}$$

where $n(C) = 0, 1$ is the Z_2 homotopy class of Q on C. Importantly, since Q is defined in all space (except at the cores of vortices), $n(C)$ is equal to the number of vortices
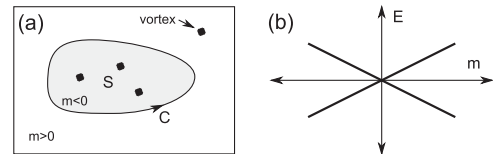

FIG. 1. (a) A 2D TI ($m < 0$) in region S with boundary C is surrounded by trivial insulator ($m > 0$). The sign of Z_{eff} depends on the number of vortices in S. (b) For $m = 0$ the eigenvalues of $\mathcal{H}^{eff}$ in (10) exhibit a linear zero crossing, which leads to a vanishing vortex fugacity.

inside the TI mod 2. This leads to a bulk characterization of the TI based on the 2D NLσM: in the TI the fugacity ν of Z_2 vortices is negative. In the trivial insulator the topological term is absent and $\nu > 0$, which is obvious for vanishing spin-orbit coupling since $e^{-S_{\rm eff}[Q]}$ is a perfect square due to spin degeneracy.

At the transition between the trivial and TI, ν must pass through zero. This suggests $\nu = 0$ at the WTI surface. To demonstrate this explicitly, we model a WTI as a layered 2D TI with helical edge modes $H = \nu_x \sigma_x k_x$ stacked in the y direction with separation a. Coupling between layers gaps the surface, except at two Dirac points at $(k_x, k_y) = (0, 0)$ and $(0, \pi/a)$. Indexing the Dirac points by $\tau_z = \pm 1$, the surface states are described by

$$H_0 = \nu_x \sigma_x k_x + \nu_y \sigma_y \tau_z k_y + m \sigma_y \tau_y. \quad (6)$$

The symmetry of the WTI under translation by one layer is described by $\exp(ip_y a) = \tau_z$. Dimerization of the layers breaks this translation symmetry, and generates a mass term $m\sigma_y \tau_y$ [19]. This is the only mass that respects TR. The topologically distinct dimerization patterns are distinguished by sgn(m). The sign reversal of the Dirac mass m also describes the low-energy theory of the 2D transition between a trivial and topological insulator [30].

Equation (3) should include a sum over vortex configurations in Q. The vortex fugacity is determined by comparing (2) in the presence and absence of vortices. Consider the simplest vortex configuration involving a single retarded and advanced pair of replicas. This can be expressed in terms of a one-parameter family of Q's of the form

$$Q(\theta) = 1_{N-1} \oplus \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \oplus 1_{N-1}. \quad (7)$$

A Z_2 vortex is then a configuration where θ winds by an odd multiple of 2π.

The Grassmann integral in (2) defines a Pfaffian, so that the vortex fugacity may be written

$$\nu = \frac{{\rm Pf}[i\sigma^y D(Q)]}{{\rm Pf}[i\sigma^y D(Q_0)]}, \quad (8)$$

where Q is a vortex configuration, and $Q_0 = \Lambda$. In the space of the two nontrivial replicas we have

$$D(Q) = (\mathcal{H}_0 - E) + i\Delta(\mu^z \cos\theta + \mu^x \sin\theta). \quad (9)$$

Here μ_z is a Pauli matrix in the space of the two nontrivial replicas. To evaluate the Pfaffian, we use a trick similar to that used by Ryu *et al.* [9] and compute $({\rm Pf}[i\sigma^y D])^2 = \det[i\sigma^y D] = \det[\mu^y D]$. This is useful because $\mu^y D \equiv \mathcal{H}^{\rm eff}$ is a Hermitian operator given by

$$\mathcal{H}^{\rm eff} = \mu^y(\mathcal{H}_0 - E) + \Delta(\mu^x \cos\theta - \mu^z \sin\theta), \quad (10)$$

so the determinant is the product of its real eigenvalues. The TR symmetry of the original $\mathcal{H}_0$ becomes a particle-hole

symmetry, $\{\mathcal{H}^{\rm eff}, \Xi\} = 0$, with $\Xi = \mu^y \sigma^y K$. Moreover, when $m = 0$, $\mathcal{H}^{\rm eff}$ decouples into two independent Hamiltonians for $\tau^z = \pm 1$. Each is identical to a topological superconductor in class D, with θ playing the role of the superconducting phase. There are two zero modes indexed by $\tau^z = \pm 1$ bound to the core of a Z_2 vortex. For $m \neq 0$, the zero modes couple and split [Fig. 1(b)]. Thus $\det[\mu^y D]$ has a second-order zero at $m = 0$, so Pf$[i\sigma^y D]$ has a first-order zero, which involves a sign change as a function of m. This shows that $\nu = 0$ for $m = 0$, so isolated Z_2 vortices are forbidden at the WTI surface. With multiple vortices the zero modes will split even for $m = 0$, leading to a nonzero Pfaffian. However, since the splitting vanishes exponentially in the separation, the vortices will be confined by a linear potential.

It is thus clear that the vortex fugacity ν is a crucial variable in the NLσM. The TI and trivial insulator involve vortex proliferation and are distinguished by sgn(ν). For $\nu = 0$, qualitatively different behavior is expected reflecting the delocalization of the WTI or TCI surfaces. For $\nu = 0$ the target space of the NLσM effectively lifts to its double cover, $\tilde{G}/\tilde{H} = SO(2N)/SO(N) \times SO(N)$, for which $\pi_1(\tilde{G}/\tilde{H}) = 0$. Since G/H and $\tilde{G}/\tilde{H}$ have identical local structure, their perturbative β functions will be identical. It is useful to first consider this behavior as a function of the replica number, N.

For $N > 1$, $\beta(t) > 0$, and the weak coupling fixed point is unstable, leading to a disordered phase even for $\nu = 0$. This phase is "less disordered" than the $\nu \neq 0$ disordered phase though. The confinement of Z_2 vortices leads to a topological order similar to a Z_2 spin liquid [31]. This can be seen by placing the system in a torus: there are four topologically disconnected sectors corresponding to the homotopy classes of $Q \in G/H$ along the two large loops. When ν is turned on in this disordered phase, the Z_2 vortices immediately condense. The $\nu = 0$ line thus describes a first-order transition between the $\nu > 0$ and $\nu < 0$ phases.

The behavior for $N \to 0$ is expected to be qualitatively different. In this case the weak coupling fixed point is stable and describes an ordered phase, which is present even for $d \leq 2$ [32]. More importantly, the arguments for the absence of localization under strong disorder presented above prove that for $\nu = 0$ the NLSM at strong coupling cannot be in a disordered phase. It is useful to consider the critical value $N = 1$ that separates these behaviors. The theory for $N = 1$ is simply the XY model, and Q is fully parametrized by θ in (7). Equation (3) becomes

$$S_{N=1} = \frac{1}{16\pi t} \int d^2 r (\nabla\theta)^2. \quad (11)$$

Since the target space, S^1, is flat, $\beta(t) = 0$ to all orders. Vortices modify the behavior. For small t, 2π vortices in θ are bound, and the system flows to a fixed line
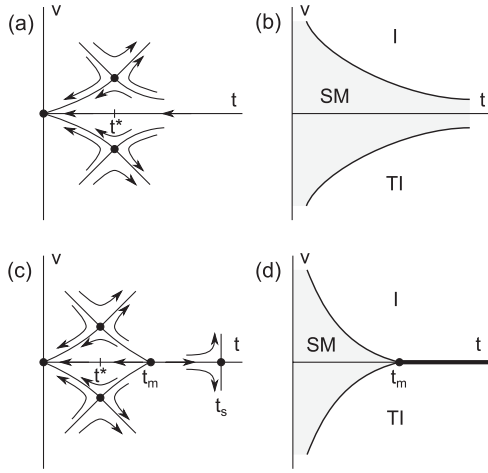
FIG. 2. (a) and (c) RG flow diagrams based on (12). The stable fixed point at $(t, v) = (0, 0)$ is the symplectic metal (SM). The unstable fixed points at $(t^*, \pm v^*)$ approach the KT transition for $\epsilon = 1 - N \ll 1$ and for $N \to 0$ are identified with the Anderson transition. (c) includes a third fixed point at $(t_m), 0$ along with a fixed point at $(t_s), 0$ describing a direct transition between TI and I. (b) and (d) are phase diagrams corresponding to (a) and (c).

parametrized by t. For $t > t^* = 1/16$ vortices unbind at a KT transition [24] to a disordered phase.

We now consider the behavior for $N < 1$, treating N as a continuous variable. Since Z_2 vortices are present for all N, it is reasonable to examine their effects as a function on N. We find that the theory can be controlled for $N = 1 - \epsilon$, with $\epsilon \ll 1$. To lowest order in ϵ and v, the KT flow equations are modified by the nonzero (but small) $\beta(t) \equiv (N - 1)\tilde{\beta}(t)$,

$$dt/d\ell = -\epsilon\tilde{\beta}(t) + v^2, \quad dv/d\ell = (2 - (8t)^{-1})v. \quad (12)$$

To this order, we are free to set the coefficient of v^2 to 1 by rescaling v. The RG flows are shown in Fig. 2. There are two fixed points at

$$t^* = 1/16, \qquad v^* = \pm[\epsilon\tilde{\beta}(t^*)]^{1/2}. \quad (13)$$

For small ϵ, these fixed points are within perturbative range of the KT fixed point. They describe a transition between the ordered and disordered phases of the $O(2N)/O(N) \times O(N)$ NLσM for $N < 1$. For $N \to 0$ we identify these fixed points with the Anderson transitions between the symplectic metal and the trivial insulator or the topological insulator. Our theory implies that these two transitions have identical bulk critical behaviors, since the total number of Z_2 vortices in a closed system is always even, so their total contribution to the partition function is always positive.

By expanding (12) about the fixed point, we can identify the critical conductivity and the correlation length exponent associated with the symplectic Anderson transition. To lowest order in ϵ we find $\sigma^* = (2\pi t^*)^{-1}e^2/h = (8/\pi)e^2/h$ and $\nu = 2t^*/[\epsilon\tilde{\beta}(t^*)]^{1/2}$. While $\tilde{\beta}(t^*)$ is not

known exactly, $\beta(t)$ has been computed perturbatively up to order t^5 [28]. The small value of t^* is well within the range of this perturbation theory. The second-order term gives only 6% correction and the higher terms are even smaller. Using the first term from (4) we find $\nu = (2/\epsilon)^{1/2}$. Extrapolating to $\epsilon = 1$ gives

$$\sigma^* \sim 2.5e^2/h, \qquad \nu \sim 1.4. \quad (14)$$

These values are rather different from numerical estimates in previous model studies, which give $\sigma^* \sim 1.4e^2/h$ and $\nu \sim 2.7$ [16,33–35], though early work on the metal to TI transition found $\nu = 1.6$ [15]. We suggest two possible origins of the discrepancy, depending on the behavior of the $N = 0$ NLσM at strong coupling, which cannot be accessed in the present analysis. One possibility is that for $N \to 0$, $\beta(t) < 0$ for all t along the line $v = 0$. The corresponding RG flow and phase diagrams are shown in Figs. 2(a) and 2(b). In this case, the symplectic metal-insulator transition is governed by the fixed point (t^*, v^*). The discrepancy in exponents is then most likely due to the slow convergence of the ϵ expansion similar to the $d = 2 + \epsilon$ expansion for the 3D Anderson transition.

A second possibility is that for $N \to 0$, $\beta(t)$ changes sign at a critical point t_m on the line $v = 0$, as hypothesized in Ref. [8] in a different context. In fact, t_m is present for $N = 1 - \epsilon$. For $N = 1$, double vortices are allowed and will in general have nonzero fugacity. The theory with both single and double vortices can be analyzed using a dual sine Gordon theory,

$$S = \int d^2r \frac{t}{\pi}(\nabla\varphi)^2 + v\cos\varphi + v_2\cos2\varphi, \quad (15)$$

where v_2 is the fugacity for double vortices. When $v = 0$, v_2 becomes relevant at $t_m = 1/4$. When v_2 flows to strong coupling, $v = 0$ describes a first-order transition similar to the case when $N > 1$. It is unlikely that this first-order transition persists to $N = 0$, which is a theory of disordered noninteracting electrons. Instead, the most likely scenario is a continuous direct transition between trivial insulator and TI controlled by a strong coupling fixed point t_s, as indicated in Figs. 2(c) and 2(d). In this scenario, while the ultimate critical behavior is controlled by the identical fixed points $(t^*, \pm v^*)$, finite size crossover effects associated with t_m, t_s could obscure the behavior.

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