Five-Loop Four-Point Amplitude of $\mathcal{N} = 4$ Super-Yang-Mills Theory

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Using the method of maximal cuts, we construct the complete D-dimensional integrand of the five-loop four-point amplitude of $\mathcal{N} = 4$ super-Yang-Mills theory, including nonplanar contributions. In the critical dimension where this amplitude becomes ultraviolet divergent, we present a compact explicit expression for the nonvanishing ultraviolet divergence in terms of three vacuum integrals. This construction provides a crucial step towards obtaining the corresponding amplitude of $\mathcal{N} = 8$ supergravity required to resolve the general ultraviolet behavior of supergravity theories.

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Recent years have seen remarkable progress in understanding and constructing scattering amplitudes in gauge and gravity theories, driven largely by the advent of on shell techniques. The advances have had broad applications including computations in quantum chromodynamics of multijet processes at the Large Hadron Collider, resummations of planar $\mathcal{N} = 4$ super-Yang-Mills (SYM) amplitudes linking them to string theory via the AdS/CFT correspondence, connections to integrability of planar $\mathcal{N} = 4$ SYM theory and studies of ultraviolet (UV) and infrared diverges in gauge and gravity theories. (See Ref. [1] for recent reviews.)

These advances have been most striking for the maximally supersymmetric $\mathcal{N} = 4$ SYM amplitude in the planar limit, where the number of color charges is large. Significant progress has also been made for the less well understood nonplanar case, which is the subject of this Letter. Nonplanar contributions to amplitudes in $\mathcal{N} = 4$ SYM theory have been obtained previously through four loops [2–7], along with detailed studies of their UV properties in higher space-time dimensions. The planar part of the five-loop amplitude is found in Ref. [8]. Here, we carry out a similar study for the five-loop four-point amplitude and analyze the UV divergences in the lowest dimension where they occur. We anticipate that our results will become useful for detailed studies of the structure of the theory, including infrared singularities, anomalous dimensions, and other observables related to amplitudes. Such studies can be a useful laboratory for quantum chromodynamics, for example, to help resolve the full structure of infrared singularities (see e.g., Ref. [9]).

Beyond the intrinsic interest for understanding $\mathcal{N} = 4$ SYM theory, our construction of the five-loop four-point amplitude is a key step towards obtaining the corresponding amplitudes of $\mathcal{N} \geq 4$ supergravity, needed to help resolve the long-standing question on the possible UV finiteness these theories. In fact, whenever a representation of an $\mathcal{N} = 4$ SYM amplitude is constructed that exhibits a duality between color and kinematics [10,11], a simple pathway exists for obtaining corresponding $\mathcal{N} \geq 4$ supergravity loop integrands [5,7,12,13]. In particular, the $\mathcal{N} = 8$ supergravity integrand follows trivially when the duality is manifest, simply by replacing color factors with the kinematic numerators of the diagrams. Although the form of the five-loop four-point amplitude presented here does not manifest the required duality, it does offer an excellent starting point for finding such representations.

Explicit constructions of amplitudes have played a key role for determining the UV divergence structure of gauge and gravity theories as a function of dimension. $\mathcal{N} = 4$ SYM theory is known [3,14] to be UV finite in dimensions

$$D < 4 + \frac{6}{L}, \qquad (L > 1), \tag{1}$$

where L is the loop order. This exhibits the well-known UV finiteness in $D = 4$ [15]. A remaining open question is whether the bound in Eq. (1) is saturated to all loop orders. From explicit computations, it is known to be saturated for $L \leq 4$ [3,4,6,7]. As commented on in Ref. [6], the $L = 5$ planar amplitude [8] also saturates the bound in Eq. (1). Below, we give a simple expression for the divergence, including nonplanar parts.

A related open question is whether maximally supersymmetric $\mathcal{N} = 8$ supergravity has the same finiteness bound as $\mathcal{N} = 4$ SYM theory, implying it is UV finite in $D = 4$, or if it has a worse behavior. (For a recent optimistic opinion see Ref. [16]; for a recent pessimistic one see Ref. [17].) Explicit calculations of the divergences [3,4,7,18,19] and symmetry and other arguments [20] show that through four loops, Eq. (1) holds in $\mathcal{N} = 8$ supergravity. Beyond this, the arguments suggest that $\mathcal{N} = 8$ supergravity will have a worse behavior, leading to a seven-loop divergence in $D = 4$. However, when similar symmetry arguments are applied to $\mathcal{N} = 4$

supergravity, they imply the existence of a valid three-loop counterterm [21]; the coefficient of this counterterm has recently been explicitly shown to vanish [13]. (See Ref. [22] for a string-based argument.) This exhibits better behavior than implied by known symmetry considerations and is in line with cancellations suggested by unitarity arguments [23]. In particular, it emphasizes the importance of directly checking the amplitudes whether Eq. (1) holds for $\mathcal{N} = 8$ supergravity at $L = 5$.

Our construction of the five-loop four-point amplitude of $\mathcal{N} = 4$ SYM theory organizes it in the form

$$\mathcal{A}_4^{(5)} = ig^{12} st A_4^{\text{tree}} \sum_{\mathcal{S}_4} \sum_{i=1}^{416} \int \prod_{j=5}^{9} \frac{d^D l_j}{(2\pi)^D} \frac{1}{S_i} \frac{C_i N_i}{\prod_{m=5}^{20} l_m^2}, \qquad (2)$$

where the second sum runs over a set of 416 distinct (nonisomorphic) graphs with only cubic (trivalent) vertices. Some sample graphs are shown in Fig. 1. The first sum runs over all 24 permutations of external leg labels indicated by $\mathcal{S}_4$. The symmetry factors S_i remove overcounts, including those arising from internal automorphism symmetries with external legs fixed. Here, we absorb all contact terms (i.e., terms with fewer than the maximum number of propagators) into graphs with only cubic vertices, by multiplying and dividing by appropriate propagators. We denote external momenta by k_i for $i = 1, \ldots, 4$ and the five independent loop momenta by l_j for $j = 5, \ldots, 9$. The remaining l_j are linear combinations of these. The color factors C_i of all graphs are obtained by dressing every three-vertex in the graph with a factor of $\tilde{f}^{abc} = \text{Tr}([T^a, T^b]T^c)$, where the gauge group generators T^a are normalized as $\text{Tr}(T^a T^b) = \delta^{ab}$. The gauge coupling is g and the crossing symmetric prefactor $st A_4^{\text{tree}}$ is in terms of the color-ordered D-dimensional tree amplitude $A_4^{\text{tree}} \equiv A_4^{\text{tree}}(1, 2, 3, 4)$ and $s = (k_1 + k_2)^2$ and $t = (k_2 + k_3)^2$.

To construct the numerators N_i, we use the method of maximal cuts [8], based on the unitarity method [24]. Application of this method and various strategies for greatly streamlining the construction of the numerators has been described in considerable detail in Ref. [6], so here we give only a brief summary. The method works in D dimensions and can be used to obtain local expressions, from which UV divergences can be straightforwardly extracted.

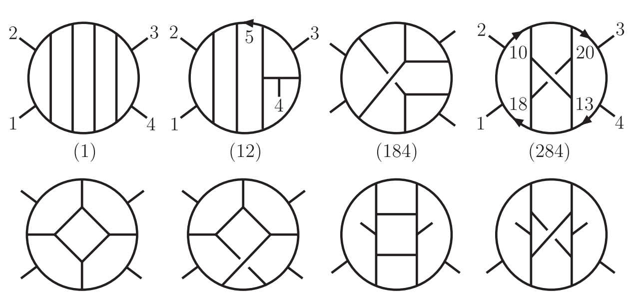

FIG. 1. Sample graphs for the five-loop four-point $\mathcal{N} = 4$ SYM amplitude. The graph labels correspond to the ones used in the Supplemental Material [27].

We start with an ansatz for the diagram numerators containing free parameters to be determined by matching against generalized unitarity cuts. Our ansatz is a polynomial of degree four in the kinematic invariants, subject to the power-counting constraint that no term has more than six powers of loop momentum. We also demand that each numerator respects the automorphism symmetries of the graph. Once a solution is found satisfying a complete set of cut conditions, we have the integrand. If an inconsistency is encountered, the ansatz must be enlarged. We note that the solutions for numerators are not unique and different choices can be mapped into each other by generalized gauge transformations [10,11,25].

The parameters of the ansatz are determined from generalized unitarity cuts that decompose a loop integrand into products of on shell tree amplitudes summed over all intermediate states, $\sum_{\text{states}} A_{(1)}^{\text{tree}} A_{(2)}^{\text{tree}} \cdots A_{(m)}^{\text{tree}}$. These cuts are organized according to the number of cut propagators that are replaced with on shell conditions. We start from the maximal cuts where all 16 internal propagators cut. After obtaining the maximal cuts, we then constructs all next-to-maximal cuts (NMCs), with 15 cut propagators. We continue this process, systematically constructing analytic expressions for (next-to)^k-maximal cuts (N^{k}MCs) with fewer and fewer imposed cut conditions. For the five-loop four-point $\mathcal{N} = 4$ SYM amplitude, this process terminates at $k = 3$, since the power counting of the theory prevents numerator factors from canceling more than 3 propagators. Representative cuts for $k = 0, 1, 2, 3$ are shown in Fig. 2. The number of nonzero (color-stripped) cuts of type N^{k}MC are 410, 2473, 7917, 15156 for $k = 0, 1, 2, 3$, respectively. This count does not include the different independent color orderings of each cut. In addition to the nonvanishing cuts, there is a large class of N$^{k\leq 3}$MCs that evaluate to zero because they contain nontrivial ($n \leq 3$)-point subamplitudes.

Each cut can be reduced to a relatively simple analytic expression. All N^{k}MCs used in the construction are evaluated in D dimensions by embedding them in auxiliary cuts that can be directly expressed in terms of simplified analytic forms. As discussed in some detail in Ref. [6], two particularly useful cuts for this purpose are two-particle cuts and box cuts. Whenever a two-particle reducible cut can be factorized into two four-point amplitudes, as

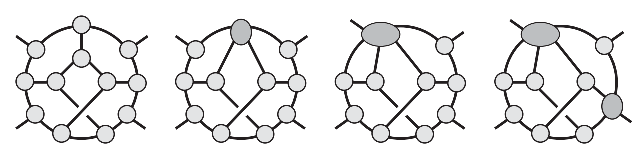

FIG. 2. Sample N^k-maximal cuts for $k = 0, 1, 2, 3$. The exposed lines are all cut.

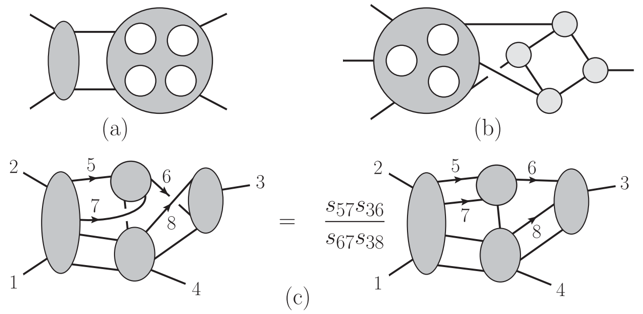

FIG. 3. Examples of simple cuts used to speed up the calculation. (a) is a two particle cut, (b) a box cut, and (c) is a sample application of the new amplitude relations of Ref. [10]. The exposed lines are all cut.

illustrated in Fig. 3(a), all contributions to the cut can be written down immediately using lower-loop results [2]. Similarly, all cut contributions that possess a four-point loop or box subdiagram, illustrated in Fig. 3(b), are simple to evaluate [6]. A third type of auxiliary cut [10], illustrated in Fig. 3(c), allows us to map known D-dimensional planar cuts to nonplanar ones via the new tree-amplitude relations uncovered in Ref. [10]. Alternatively, one can construct numerator representations that obey the color-kinematics duality for each cut separately [6], giving local numerator relations between planar and nonplanar diagrams, up to terms that vanish on the cut. This technique is especially useful whenever the cut contains massless bubble or tadpole subdiagrams (as sometimes occurs for N^3MCs), since the local numerators are automatically free of spurious singularities that can appear with other methods. A fourth type of auxiliary cut valid in D dimensions and used in our construction is one that maps five-loop nonplanar cuts to simpler six-loop planar cuts [26].

We have found a choice of parameters in the starting ansatz whose cuts correctly reproduce the N^kMCs at the level of the integrand. We, thus, have a complete integral representation of the five-loop amplitude. As a few simple examples, the numerators of graphs 1, 12, and 284 are

$$\begin{aligned} N_1 &= s^4, \\ N_{12} &= 2s^3 k_3 \cdot l_5, \\ N_{284} &= 2s^2[(l_{10} \cdot l_{20})^2 + (l_{13} \cdot l_{18})^2], \end{aligned} \tag{3}$$

corresponding to the graphs in Fig. 1 labeled as (1), (12), and (284) and matching the labeling in the Supplemental Material [27]. The lines with arrows in Fig. 1 give the momentum labels and directions. The symmetry factors $1/S_i$ for these graphs are, respectively, 1/4, 1, and 1/4.

The complete set of 416 nonvanishing graphs with their associated symmetry factors, numerators, and color factors are included in the Supplemental Material [27]. We note that graphs 61, 67, 133, 137, 263, 382, 412 have vanishing color factors for a general gauge group (due to symmetry properties of the graph), and hence, do not contribute to the amplitude. However, we include them in our representation because they are needed for constructing gravity amplitudes [6].

We have carried out extensive cross checks on our result. The cut construction automatically cross checks the vast majority of contributions because they are detected in multiple independent channels. As an additional rather nontrivial check, in four dimensions, we confirmed numerically that all the analytically-obtained cuts are correct; to carry out this check, we used the simple algorithms of Ref. [28] for carrying out the supersums appearing in the cuts. We have also carried out systematic cross checks using generalized cuts with up to six collapsed propagators. Furthermore, we have evaluated a set of cuts that suffices to detect all "snail" contributions, equivalent to bubbles on external legs (see sections 2D and 3C of Ref. [7]), showing that such contributions do not appear in our representation.

Starting with the constructed integrand, we obtain the potential logarithmic divergence in the five-loop critical dimension, $D = 26/5$, following the same strategy as at lower loops [6,7,18,19]: We expand the amplitudes at small external momentum and keep the leading term. The result of this expansion is a sum of about 185 vacuum diagrams; a few of which are displayed in Fig. 4. As discussed in Refs. [7,19], the vacuum integrals in this expansion are not all independent (so the precise number appearing initially can vary). We derive consistency relations between the vacuum integrals by considering auxiliary linearly divergent integrals of similar propagator structure, expanding them around zero external momenta, and requiring that the results of the expansion be independent of different integrand parametrizations. This also directly cross checks the procedure for integral reduction since we obtain a

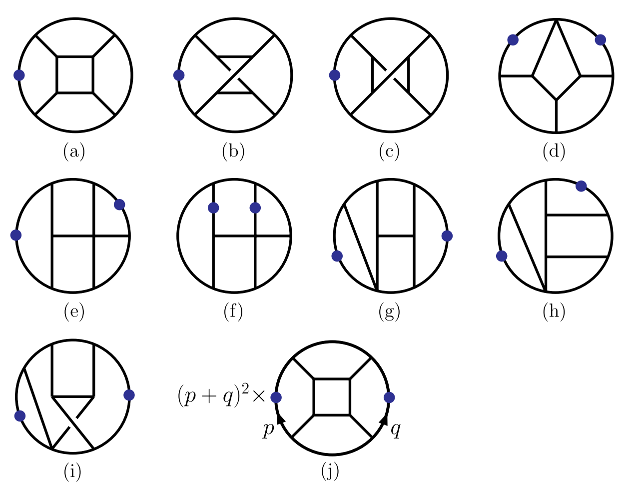

FIG. 4 (color online). Some of the five-loop vacuum integrals that appear in intermediate steps. Only (a), (b), and (c) appear in the final UV divergence. Integral (j) has a nontrivial numerator factor, as indicated. The (blue) dots indicate that a propagator is squared.

highly overconstrained set of homogeneous consistency equations. The fact that no positive definite integral is set to zero by this system is a strong check on the calculation. These consistency relations eliminate most of the vacuum diagrams. Two examples are,

$$\begin{aligned} V^{(j)} &= \frac{24}{5}V^{(a)} - 2V^{(d)}, \\ V^{(b)} &= 2V^{(c)} + 35V^{(i)} + \frac{365}{6}V^{(d)} - \frac{4175}{162}V^{(e)} \\ &\quad - \frac{1045}{18}V^{(f)} - \frac{9865}{81}V^{(g)} + \frac{305}{3}V^{(h)}, \end{aligned} \tag{4}$$

where the labels correspond to the ones in Fig. 4.

After using the consistency relations, the leading UV divergence is remarkably simple and given by only three vacuum integrals. For $SU(N_c)$, it is

$$\begin{aligned} \mathcal{A}_4^{(5)}|_{\rm div} &= \frac{144}{5} g^{12} st A_4^{\rm tree} N_c^3 \\ &\quad \times [N_c^2 V^{(a)} + 12(V^{(a)} + 2V^{(b)} + V^{(c)})] \\ &\quad \times (t\tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4} + s\tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}). \end{aligned} \tag{5}$$

With the chosen normalization, the Wick-rotated vacuum integrals in Eq. (5) are all positive definite, proving that no further hidden cancellations remain at $L = 5$ in the critical dimension for either leading- or subleading-color contributions. Using FIESTA [29], we have numerically evaluated the integrals giving,

$$V^{(a)} = \frac{0.331K}{\epsilon}, \qquad V^{(b)} = \frac{0.310K}{\epsilon}, \qquad V^{(c)} = \frac{0.291K}{\epsilon}, \tag{6}$$

where the dimensional regularization parameter is $\epsilon \equiv (26/5 - D)/2$, $K = 1/(4\pi)^{13}$ and numerical integration uncertainties are below the displayed digits. It is interesting that the ratio between the subleading and leading contributions $45.0/N_c^2$ is rather close to the three- and four-loop ratios, $43.3/N_c^2$ and $44.4/N_c^2$ [6]. A striking feature of the result [Eq. (5)] is that the divergence does not contain terms beyond $\mathcal{O}(1/N_c^2)$ suppression, nor does it contain double-trace contributions when converted to an $SU(N_c)$ color-trace representation, in line with expectations from lower loops [6]. The second of these features has already been discussed in Refs. [6,30]. Furthermore, the three integrals and their relative coefficients have a remarkable similarity with the corresponding ones at four loops, as seen by comparing to Eq. (5.33) of Ref. [6]. At lower loops, exactly the same combination of integrals appearing in the subleading-color contributions to the $\mathcal{N} = 4$ SYM divergences appears in the corresponding ones of $\mathcal{N} = 8$ supergravity [6]. A natural conjecture is that the same holds at five loops, so that the two theories share the same critical dimension, $D = 26/5$.

In summary, the five-loop amplitude we have constructed here offers detailed information on the structure of the nonplanar sector of $\mathcal{N} = 4$ SYM theory. As a first application, we have shown that simple patterns for divergences in the dimension where they first appear continue to hold through five loops; this hints that the divergences are controlled by a deep structure of the theory. Our construction of the five-loop four-point amplitude is an excellent starting point to try to find a representation exhibiting the duality between color and kinematics. We expect that the results presented here will be crucial input for obtaining corresponding supergravity amplitudes and for studying their UV behavior.

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