Ultralight Fractal Structures from Hollow Tubes

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A fractal design is shown to be highly efficient both as a load bearing structure and as a general metamaterial. Through changing the hierarchical order of the structure, the scaling of material required for stability against loading can be manipulated. We show that the transition from solid to hollow beams changes the scaling in a manner analogous to increasing the hierarchical order by one. An example second order solid beam frame is constructed using rapid prototyping techniques. The optimal hierarchical order of the structure is found for different values of loading. Possible fabrication methods and applications are then discussed.

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Introduction.—Hierarchical designs are found throughout nature as highly efficient materials and structures under diverse loading conditions [1]. One particular example is the trabecular bone which serves to pair strength and stiffness requirements with minimal weight cost [2,3]. Recent theoretical work on hierarchical and fractal geometries has shown that the scaling of material required for stability against loading can be altered by changing the hierarchical order of a structure [4–7]. The fractal dimension of such a structure has also been obtained and shown to have a nontrivial relationship with the loading for which the structure is optimized [4].

With the increased use of novel fabrication methods, a material's architecture can be controlled over an ever increasing set of length scales. Manipulation of a material's properties can then be achieved through the prudent choice of design parameters. For example, the Poisson's ratio and the elastic modulus of a second order hierarchical honeycomb design can be altered through the choice of geometric structural parameters [8]. Furthermore, these structures are found to have high strength relative to density when compared to more conventional designs [1,8,9]. References [10–12] show similar principles for second order sandwich core structures.

Recent works [13] produced complex architectures by using a photopolymer lattice as a scaffold for creating hierarchical, ultralight ($< 10\ \mathrm{mg\,cm^{-3}}$) metallic structures from hollow tubing. With this technique, a tubular lattice was created with features on length scales from nano- to centimeters. In the present Letter, such architectures are generalized to incorporate any degree of hierarchy through a self-similar design principle. We show that varying the order of hierarchy serves to change the scaling of material required to make a stable structure against the applied load. We find the transition from solid [5] to hollow tube construction equivalent (in scaling properties) to increasing the order of structural hierarchy by one. An example space frame constructed using solid beams is produced through rapid prototyping technologies [14]. Our work shows the plausibility of the hollow tube design for use in general as a hierarchical space frame and an ultralight metamaterial.

Generation-0.—As a reference, we first consider the amount of material that is required to construct a simply supported beam of length L, stable under a compressive load F. We define the nondimensional loading and volume parameters,

$$f \equiv \frac{F}{YL^2}, \qquad v \equiv \frac{V}{L^3}, \tag{1}$$

where Y is the Young's modulus of the material and V is the volume of the structure. In all realistic applications, both of these nondimensional parameters are much smaller than 1. If the beam is made up of solid material the only restriction to loading is given by Euler buckling [15],

$$F < \frac{\pi^2 YI}{L^2}, \tag{2}$$

where I is the second moment of area. Given a circular beam cross section ($I = \pi r^4/4$) it is straightforward to show

$$v \sim f^{1/2}. \tag{3}$$

If instead, we take the circular beam to be hollow, we would have two restrictions on loading: first, that of Eq. (2) with $I = \frac{1}{4}\pi[(r+t)^4 - r^4]$, where t is the shell thickness and r the beam radius; and second, we would have to consider a short wavelength deformation, or Koiter buckling [16], which gives us a second inequality,

$$F < \frac{2\pi Y t^2}{\sqrt{3(1-\nu^2)}}, \tag{4}$$

where ν is Poisson's ratio. Setting the geometry of the beam to be such that both failure modes occur at the same loading value, we see that for the hollow beam [7],

$$v \sim f^{2/3}. \tag{5}$$

In the regime $f \ll 1$ this change in scaling law represents a saving in material over the solid beam. It is the hollow beam that will be termed the generation-0 design in this work.

Scaling for hollow generation-1 structure.—The generation-1 structure is a simple space frame made up of n octahedra and two end tetrahedra constructed from hollow beams as shown in Fig. 1(a). Defining the length of the whole structure to be L, and the length of a constituent beam to be L_0, we find that

$$L = \sqrt{2/3}(n + 2)L_0. \tag{6}$$

Assuming all beams in the structure to be made up of identical beams which perform in a Hookean manner prior to Euler buckling and whose spring constant is

$$k_0 = \frac{YA}{L_0}, \tag{7}$$

where A is the cross-sectional area of the beam, the whole frame can be seen to have an effective bending stiffness (YI) and spring constant given by

$$YI = BL_0^3 k_0, \qquad k = \frac{36k_0}{11n + 43}, \tag{8}$$

respectively, where B is a constant found to be $B = 0.254 \pm 0.001$ [5]. All joints are taken to be freely hinged. At the ends of the structure, different boundary conditions may apply for the global Euler buckling, without changing the scaling behavior [14]. If we orient the structure such that the end points of the tetrahedra are aligned along the z axis in Cartesian coordinates and load these end points with a force, F, in a compressive manner, we find that all beams parallel with the x-y plane are under tension. Assuming $n \geq 2$, the beams under tension making up the end tetrahedra support a load of $\frac{F}{2\sqrt{6}}$ while other tension members support a load of $\frac{F}{3\sqrt{6}}$. Furthermore, we find all other beams support a compressive load, and the beams connected to the end points are acted on by a force of

$$F_0 = \frac{F}{\sqrt{6}}, \tag{9}$$

while, all other beams under compression take half this loading each. Defining

$$f_0 \equiv \frac{F_0}{YL_0^2}, \tag{10}$$

and using Eqs. (2), (4), (9), and (10) we see that if

$$t = \kappa_t L_0 f_0^{1/2}, \qquad r = \kappa_r L_0 f_0^{1/6}, \tag{11}$$

all beams under loading F_0 are set to be on the point of both Koiter and Euler buckling (κ_r and κ_t are independent of loading). Then, using Eqs. (6)–(11) and taking the space frame to be on the point of buckling due to the most vulnerable failure mode, we obtain

$$n = -2 + \left\lfloor \frac{6^{1/4}\pi^{5/6}B^{1/2}[3(1-\nu^2)]^{1/12} f_0^{-(1/6)}}{2^{2/3}} \right\rfloor, \tag{12}$$

where $\lfloor\cdot\rfloor$ is the floor function. Then, retaining the definitions of v and f given previously, we arrive at

$$f = \frac{3\sqrt{6}}{2}(n + 2)^{-2} f_0, \tag{13}$$

$$v = 27\sqrt{6}\frac{(n + 1)f_0^{2/3}[3(1-\nu^2)]^{1/6}}{\pi^{1/3}2^{4/3}(n + 2)^3}, \tag{14}$$

and through the use of Eq. (12) and elimination of f_0,

$$v \sim f^{3/4}. \tag{15}$$

Generation-n optimization.—The generation-n structure can be created through an iterative procedure: in the generation-1 structure, the simple beam that makes up the generation-0 structure is replaced with a space frame. It is an analogous step that takes us from the generation-1 structure to the generation-2 structure: all simple beams in the structure under compression are replaced by (scaled) generation-1 frames. An example generation-2 structure is shown in Fig. 1(b) and an example structure constructed through a rapid prototyping procedure is shown in Fig. 2.

For a given property of the structure that is recurrent on different levels of the structure, $X_{G,i}$ will denote the property X on the ith level of a generation G structure, where $i = 0$ and $i = G$ denote the shortest and longest length scales respectively. Thus $L_{G,G} \equiv L$ is the length of the whole generation-n structure and $L_{G,0} \equiv L_0$ is that of the smallest beams; $F_{G,G} \equiv F$ is the loading on the whole space frame and $F_{G,0} \equiv F_0$ is the loading on the smallest (end) compressively loaded components. It can be shown that the properties of the frame are given by [5]

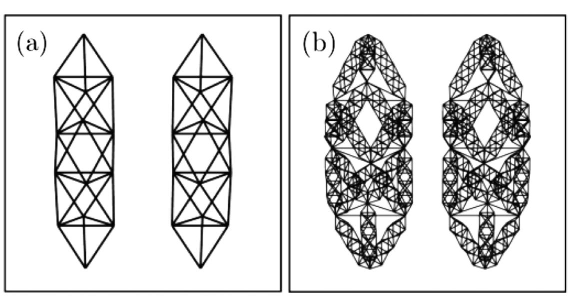

FIG. 1. Generation-1 and generation-2 frames shown labeled (a) and (b) respectively. Images are stereographic: to view a single three-dimensional structure hold the figure 20 cm away and focus ‘through‘ the page until both images merge.

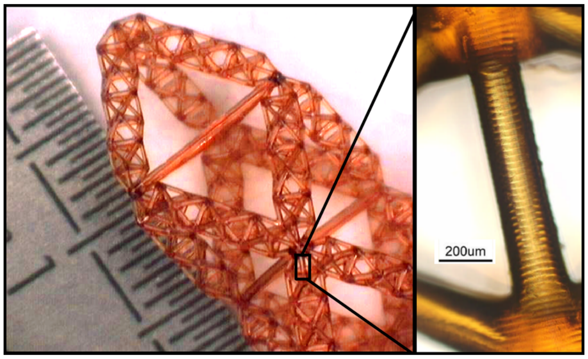

FIG. 2 (color online). The end tetrahedron and octahedron of a generation-2 structure constructed through a rapid prototyping procedure; this structure is constructed using solid beams. In principle, this could be used as a scaffold [13] to construct a hollow, metallic structure of similar geometry. Inset shows the layered nature of the material in a single beam, with a layer thickness of 25 μm.

$$L_{G,i} = \sqrt{2/3}(n_{G,i} + 2)L_{G,i-1}, \tag{16}$$

$$(YI)_{G,i} = BL^3_{G,i-1}k_{G,i-1} \tag{17}$$

$$k_{G,i} = \frac{36k_{G,i-1}}{11n_{G,i} + 43}, \tag{18}$$

$$F_{G,i} = \sqrt{6}F_{G,i-1}, \tag{19}$$

where $k_{G,i}$ is the effective spring constant of all (sub) structures of length $L_{G,i}$, and $F_{G,i}$ is the greatest compressive load experienced by a substructure of length $L_{G,i}$. We see that to avoid Euler buckling at each hierarchical length scale, we must impose the constraint,

$$F_{G,i} < \frac{\pi^2(YI)_{G,i}}{L^2_{G,i}}, \tag{20}$$

for all i. Given that the smallest beams are made of hollow tubes, we have still to take into account the possibility of Koiter buckling, and this constraint on loading provides us with the inequality stated in Eq. (4).

Then, defining the geometry such that Euler buckling and the short wavelength Koiter buckling occur simultaneously in the beams of length $L_{G,0}$, through the use of Eqs. (4), (10), and (20) with $i = 0$, it can be shown that the expressions for t and r given in Eq. (11) are still valid for the higher generation structures. Then using Eqs. (10), (11), and (16)–(20) and setting all (sub) frames to be on the point of failure due to Euler buckling, we find that the value for $n_{G,1}$ is equal to the value of n in Eq. (12), and, for $i > 1$ we have,

$$n_{G,i} = -2 + \left\lfloor \left\{ \frac{\sqrt{6}}{2^{4/3}} \pi^{5/3} B[3(1-\nu^2)]^{1/6} f_0^{-(1/3)} \right.\right.$$
$$\left.\left. \times\, 12^{i-1} \prod_{j=1}^{i-1} \frac{n_{G,j} + 2}{11n_{G,j} + 43} \right\}^{1/2} \right\rfloor. \tag{21}$$

For $G > 1$, we see that

$$f = \left(\frac{27}{2}\right)^{G/2} f_0 \prod_{j=1}^{G} (n_{G,j} + 2)^{-2}, \tag{22}$$

$$\upsilon = \left(\frac{9\sqrt{6}}{2}\right)^G \frac{f_0^{2/3}[3(1-\nu^2)]^{1/6}}{2^{1/3}\pi^{1/3}} \prod_{k=1}^{G} \frac{n_{G,k} + 1}{(n_{G,k} + 2)^3}$$
$$\times \left[3 + \sum_{q=1}^{G-1} 4^q \prod_{j=1}^{q} \frac{(n_{G,j} + 2)^2}{(11n_{G,j} + 43)(n_{G,j} + 1)} \right], \tag{23}$$

where, to obtain the former equation, Eqs. (1), (10), (16), and (19) were used, and in the latter, Eqs. (1), (11), and (16). The scaling of material required to make a stable structure out of hollow tubes, to leading order, is therefore

$$\upsilon = \kappa(G)f^{(G+2)/(G+3)}, \tag{24}$$

which, for $f \ll 1$, shows a gain in scaling efficiency over the structure previously described in Ref. [5] equivalent to raising the generation of the structure by one. These scalings are demonstrated in Fig. 3 for various values of loading. It is possible to optimize for different Young's moduli of individual beams (or beams of different radii), which improves the prefactor of our results but the scaling power remains unchanged.

Figure 4 illustrates the material saving of the hierarchical frame of optimal generation over a solid beam construction for material properties close to those of steel ($Y = 210$ GPa, $\nu = 0.29$ and density, $\rho = 8000$ kg m^{-3}). For example, to construct a space frame to withstand a load of $F = 10$ kN, over a distance of $L = 200$ m, a solid beam would require 79 t of material while the optimal hollow, hierarchical frame would require just 162 kg or 0.2% of the solid beam's weight.

Conclusions and discussion.—An elastically isotropic, rigid metamaterial can be created by joining the nearest and next-nearest neighbors of a fcc lattice of points with generation-G hierarchical beams. Through changing the generation of this metamaterial, it is seen through Eq. (24),

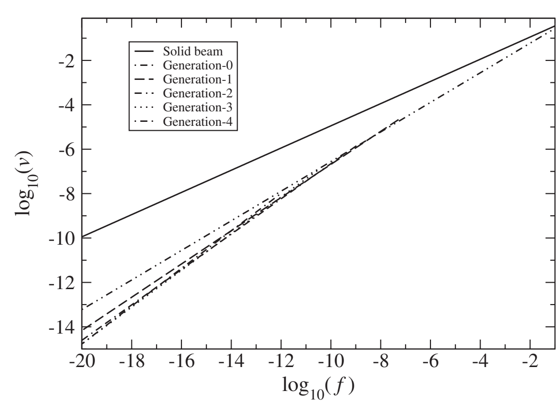

FIG. 3. Volume required for a stable structure against loading for which the structure is optimized, showing generations 0–4.

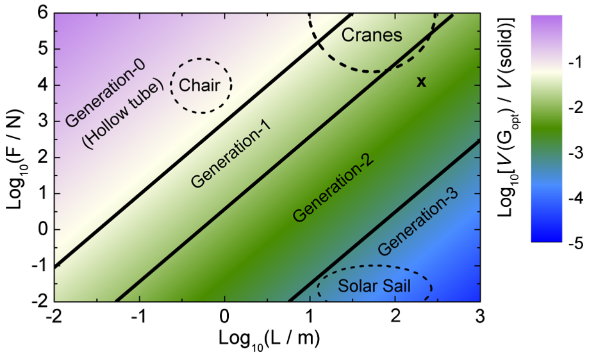

FIG. 4 (color online). The material saving for a given F and L over a solid beam assuming material properties close to those of steel. Also depicting the progression of optimality from simple structures (chair legs) to structures with many levels of hierarchy. Also shown is the point $F = 10$ kN, $L = 200$ m ($\times$) and the regions appropriate for construction of a solar sail compression beam and the boom of a crane [20].

that the scaling of volume fraction of real material against crush pressure can be altered in a systematic, advantageous manner.

According to this formulation, an optimized version of the lattice presented in Ref. [13] would represent a generation-0 metamaterial. The construction of higher order frames has shown that it is possible to construct the intricate detail of a second order frame on the centimeter scale. The process of electroless deposition combined with etching would bring about a hollow generation-2 structure of the same scale.

Finite element analysis and mechanical testing have been undertaken on the frames described here [14]. The failure modes predicted by finite element analysis match those predicted in this theory while the mechanical testing shows a high dependence on the construction material properties. Tensional failure may be limiting for brittle materials, but becomes rapidly less important as $f \to 0$.

We have presented a design for a fractal structure and shown it to be highly efficient under compressive loading. Analyzing both short and long wavelength deformations on the smallest length scales and Euler buckling at all other hierarchical levels, we have optimized the design presented and given bounds for material usage against force withstood. We have shown the scaling of material required against that force can be manipulated in a beneficial manner though changing the order of hierarchy. We found that the hollow tube construction results in an alteration of scaling equivalent to increasing the order of hierarchy by one over the solid beam design. We have fabricated an example generation-2 frame on the centimeter to millimeter scale, with micrometer layer thickness, and have shown the design is plausible for implementation.

These designs show considerable potential: the metamaterial presented above is a realistic basis for objects of any shape, and with ongoing advances in rapid prototyping technologies and three-dimensional printing the widespread adoption of such designs is not impossible. The degree to which the mechanical properties can be tailored through the choice of geometric parameters at different length scales, allied with their high strength to weight ratio make hierarchical structures of the kind analyzed here interesting for a variety of use cases. It is in the regime of low loading and large lengths that hierarchical designs are at their most advantageous. In space engineering, these conditions often apply [17], while low mass is an important design criterion. The construction of solar sails, for example, requires long compression members of minimal mass [18]. Typical parameters for their construction are shown in Fig. 4, and high efficiency is found in this region.

Finally it is noted that the smallest possible building blocks for these designs would be single- and multiwalled carbon nanotubes. It has been shown that both Koiter and Euler buckling modes are closely approximated by Eqs. (2) and (4) (up to a prefactor) for these [19], and thus the analysis presented here is expected to still hold.

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