Measuring Gaussian Quantum Information and Correlations Using the Rényi Entropy of Order 2

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We demonstrate that the Rényi-2 entropy provides a natural measure of information for any multimode Gaussian state of quantum harmonic systems, operationally linked to the phase-space Shannon sampling entropy of the Wigner distribution of the state. We prove that, in the Gaussian scenario, such an entropy satisfies the strong subadditivity inequality, a key requirement for quantum information theory. This allows us to define and analyze measures of Gaussian entanglement and more general quantum correlations based on such an entropy, which are shown to satisfy relevant properties such as monogamy.

In quantum information theory, the degree of information contained in a quantum state ρ is conventionally quantified via the von Neumann entropy of the state $\mathcal{S}(\rho) = -\operatorname{tr}(\rho \ln \rho)$, that is *the* direct counterpart to the Shannon entropy in classical information theory [1]. The most fundamental mathematical implications and physical insights in quantum information theory, ranging from the Holevo bound [2] all the way to the whole genealogy of quantum communication protocols [3,4], rely on a key property satisfied by the von Neumann entropy, the strong subadditivity inequality [5,6]

$$\mathcal{S}(\rho_{AB}) + \mathcal{S}(\rho_{BC}) \geqq \mathcal{S}(\rho_{ABC}) + \mathcal{S}(\rho_B), \qquad (1)$$

for an arbitrary tripartite state ρ_{ABC}. The strong subadditivity inequality implies in particular that the mutual information

$$\mathcal{I}(\rho_{A:B}) = \mathcal{S}(\rho_A) + \mathcal{S}(\rho_B) - \mathcal{S}(\rho_{AB}), \qquad (2)$$

which measures the total correlations between the subsystems A and B in the bipartite state ρ_{AB}, is always nonnegative. However, in classical as well as quantum information theory, several other entropic quantities have been introduced and studied. In particular, Rényi-α entropies [7] are an interesting family of additive entropies, whose interpretation is related to derivatives of the free energy with respect to temperature [8], and which have found applications especially in the study of channel capacities [9], work value of information [10], and entanglement spectra in many-body systems [11]. They are defined as

$$\mathcal{S}_\alpha(\rho) = (1 - \alpha)^{-1} \ln \operatorname{tr}(\rho^\alpha), \qquad (3)$$

and reduce to the von Neumann entropy in the limit $\alpha \to 1$. Rényi entropies are powerful quantities for studying quantum correlations in multipartite states. For a bipartite pure state ρ_{AB}, any of the entropies in Eq. (3) evaluated on the reduced density matrix of one subsystem only, say ρ_A, is an entanglement monotone [12], dubbed Rényi-α

entanglement [13]. Any such measure can be extended to mixed states via conventional convex roof techniques [14]. Using the $\alpha = 2$ instance, $\mathcal{S}_2(\rho) = -\ln \operatorname{tr}(\rho^2)$, the ensuing Rényi-2 measure of entanglement has been defined and proven to satisfy the important 'monogamy' inequality [15,16] for multiqubit states [13,17]. Nonetheless, Rényi-α entropies for $\alpha \neq 1$ are not in general subadditive [1]; this entails, e.g., that if one replaces $\mathcal{S}$ by $\mathcal{S}_{\alpha \neq 1}$ in (2), the corresponding quantity can become negative, i.e., meaningless as a correlation measure; see Ref. [18] for explicit two-qubit instances of this fact when $\alpha = 2$.

In this Letter, we focus our attention on multimode quantum harmonic oscillators, and analyze the informational properties of general Gaussian states as measured by the Rényi-2 entropy. Gaussian states constitute versatile resources for quantum communication protocols with continuous variables [19,20] and are important test beds for investigating the structure of quantum correlations [21], whose role is crucial in fields as diverse as quantum field theory, solid state physics, quantum stochastic processes, and open system dynamics. Gaussian states naturally occur as ground or thermal equilibrium states of any physical quantum system in the 'small-oscillations' limit [22,23], and can be very efficiently engineered, controlled and detected in various experimental setups, including light, atomic ensembles, trapped ions, nanomechanical or optomechanical resonators, and hybrid interfaces thereof [20]. Given the special role played by Gaussian states and operations, which are formally 'extremal' with respect to several quantum primitives [23], a Gaussian-only theory of quantum information is actively pursued [24,25]. In particular, a key conjecture which—despite its disproof for general channels [26]—still stands within the Gaussian scenario, is the so-called minimum output entropy conjecture for bosonic channels [24,25,27,28]. Interestingly, the conjecture is verified (for all phase-insensitive channels) if using Rényi-α entropies with $\alpha \geqq 2$ [29], but to date is resisting an analytical proof for $\alpha \to 1$ [30]. This somehow raises the question whether the von Neumann

entropy is indeed the most natural one within the Gaussian scenario [31].

Here we show that the Rényi-2 entropy should be regarded as a specially meaningful choice to develop a Gaussian theory of quantum information and correlations. We prove that the Rényi-2 entropy $\mathcal{S}_2$ satisfies the strong subadditivity inequality (1) for all Gaussian states, is operationally linked to the Shannon entropy of Gaussian Wigner distributions (providing a natural measure of state distinguishability in phase space), and can be employed to define valid measures of Gaussian entanglement [32,33] and general discordlike quantum correlations [34–38], whose monogamy properties [15,16,39–42] we investigate in detail. In particular, we obtain a truly *bona fide* measure of genuine tripartite entanglement for three-mode Gaussian states, based on $\mathcal{S}_2$. Our study allows us to explore within a unified framework the various facets of nonclassicality in the Gaussian realm.

Rényi-2 entropy for Gaussian states.—We consider an n-mode continuous variable system; we collect the quadrature operators in the vector $\hat{\boldsymbol{R}} = (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \ldots, \hat{q}_n, \hat{p}_n)^{\mathsf{T}} \in \mathbb{R}^{2n}$ and write the canonical commutation relations compactly as $[\hat{R}_j, \hat{R}_k] = i(\boldsymbol{\omega}^{\oplus n})_{j,k}$ with

$$\boldsymbol{\omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

being the symplectic matrix. A Gaussian state ρ is described by a positive, Gaussian-shaped Wigner distribution in phase space,

$$W_\rho(\boldsymbol{\xi}) = \frac{1}{\pi^n \sqrt{\det\boldsymbol{\gamma}}} \exp(-\boldsymbol{\xi}^{\mathsf{T}} \boldsymbol{\gamma}^{-1} \boldsymbol{\xi}), \tag{4}$$

where $\boldsymbol{\xi} \in \mathbb{R}^{2n}$, and $\boldsymbol{\gamma}$ is the covariance matrix (CM) of elements $\gamma_{j,k} = \mathrm{tr}[\rho\{\hat{R}_j, \hat{R}_k\}_+]$, which (up to local displacements) completely characterizes the Gaussian state ρ.

Let us now evaluate different entropic quantities on the Gaussian state ρ. All the measures defined below are invariant under local unitaries, so we will assume our states to have zero first moments, $\langle\hat{\boldsymbol{R}}\rangle = \boldsymbol{0}$, without loss of generality. The purity is easily computed as $\mathrm{tr}\rho^2 = (2\pi)^n \int_{\mathbb{R}^{2n}} W_\rho^2(\boldsymbol{\xi})d^{2n}\boldsymbol{\xi} = (\det\boldsymbol{\gamma})^{-(1/2)}$. Hence, the Rényi-2 entropy of an arbitrary n-mode Gaussian state is

$$\mathcal{S}_2(\rho) = \frac{1}{2} \ln(\det\boldsymbol{\gamma}), \tag{5}$$

ranging from 0 on pure states ($\det\boldsymbol{\gamma} = 1$) and growing unboundedly with increasing mixedness (i.e., temperature) of the state. The von Neumann entropy is instead a more complicated function that depends on the local temperatures of all the n normal modes of ρ, that is, on the full symplectic spectrum of $\boldsymbol{\gamma}$ [21,31,43,44]. On the other hand, since the Wigner function is a valid probability distribution providing a fully equivalent description of ρ, we can

alternatively compute the Shannon entropy of W_ρ to obtain an alternative quantifier of the informational content of any Gaussian state. Such an entropy has a clear interpretation in terms of phase-space sampling via homodyne detections [45]. The continuous (Boltzmann-Gibbs-)Shannon entropy for a probability distribution $P(x)$ is defined as $H(P) = -\int P(x)\ln P(x)dx$ [46]. It can be shown that [47] (see the Supplemental Material [48] for a derivation)

$$H(W_\rho) = -\int_{\mathbb{R}^{2n}} W_\rho(\boldsymbol{\xi}) \ln[W_\rho(\boldsymbol{\xi})]d^{2n}\boldsymbol{\xi}$$
$$= \mathcal{S}_2(\rho) + n(1 + \ln\pi). \tag{6}$$

Interestingly, we see that this sampling entropy coincides (modulo the additional constant) with the Rényi-2 entropy rather than the von Neumann one for Gaussian states. We can now take an extra step and introduce a 'relative sampling entropy', to quantify the phase-space distinguishability of two n-mode Gaussian quantum states ρ_1 and ρ_2 (with CMs $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$, respectively), defined as the relative Shannon entropy (also known as Kullback-Leibler divergence [1,6]) between their respective Wigner distributions W_{ρ_1} and W_{ρ_2}, yielding [48]

$$H(W_{\rho_1} \| W_{\rho_2}) \doteq \int_{\mathbb{R}^{2n}} W_{\rho_1}(\boldsymbol{\xi}) \ln\left(\frac{W_{\rho_1}(\boldsymbol{\xi})}{W_{\rho_2}(\boldsymbol{\xi})}\right)d^{2n}\boldsymbol{\xi}$$
$$= \frac{1}{2}\left[\ln\left(\frac{\det\boldsymbol{\gamma}_2}{\det\boldsymbol{\gamma}_1}\right) + \mathrm{tr}(\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2^{-1})\right] - n. \tag{7}$$

Let us evaluate Eq. (7) when $\rho_1 \equiv \rho_{AB}$ is a generic Gaussian state of a composite system, partitioned into two subsystems A and B (of n_A and n_B modes respectively, with $n_A + n_B = n$), and $\rho_2 \equiv \rho_A \otimes \rho_B$ is the tensor product of the two marginals of ρ_{AB}. Writing the CMs in block form,

$$\boldsymbol{\gamma}_1 \equiv \boldsymbol{\gamma}_{AB} = \begin{pmatrix} \boldsymbol{\gamma}_A & \boldsymbol{s}_{AB} \\ \boldsymbol{s}_{AB}^{\mathsf{T}} & \boldsymbol{\gamma}_B \end{pmatrix}, \qquad \boldsymbol{\gamma}_2 \equiv \boldsymbol{\gamma}_A \oplus \boldsymbol{\gamma}_B, \tag{8}$$

we have $\mathrm{tr}(\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2^{-1}) = 2(n_A + n_B) = 2n$, which entails

$$H(W_{\rho_{AB}} \| W_{\rho_A \otimes \rho_B}) = H(W_{\rho_A}) + H(W_{\rho_B}) - H(W_{\rho_{AB}})$$
$$= \frac{1}{2} \ln\left(\frac{\det\boldsymbol{\gamma}_A \det\boldsymbol{\gamma}_B}{\det\boldsymbol{\gamma}_{AB}}\right)$$
$$= \mathcal{S}_2(\rho_A) + \mathcal{S}_2(\rho_B) - \mathcal{S}_2(\rho_{AB})$$
$$\doteq I_2(\rho_{A:B}). \tag{9}$$

The above equation defines the 'Gaussian Rényi-2 (GR2) mutual information' I_2 for an arbitrary bipartite Gaussian state ρ_{AB}. It follows that $I_2(\rho_{A:B}) \geq 0$, as it coincides exactly with the Shannon continuous mutual information of the Wigner function of ρ_{AB}, which is positive semidefinite. The expression in Eq. (9), which is analogous to the more familiar von Neumann one [Eq. (2)], has, thus, a precise operational interpretation as the amount of extra discrete information (measured in natural bits) that needs

to be transmitted over a continuous variable channel to reconstruct the complete joint Wigner function of ρ_{AB} rather than just the two marginal Wigner functions of the subsystems [45,46]; in short, $I_2(\rho_{A:B})$ measures the total quadrature correlations of ρ_{AB}.

We can then enquire whether a more general property such as the strong subadditivity inequality (1)—which would independently imply the non-negativity of I_2— holds in general for $\mathcal{S}_2$ in the Gaussian scenario. Let ρ_{ABC} be a tripartite Gaussian state whose subsystems encompass an arbitrary number of modes. Writing its CM in block form as in Eq. (8), and using the definition (5), we have the following.

Theorem 1.—The Rényi-2 entropy $\mathcal{S}_2$ satisfies the strong subadditivity inequality for all Gaussian states ρ_{ABC},

$$\mathcal{S}_2(\rho_{AB}) + \mathcal{S}_2(\rho_{BC}) - \mathcal{S}_2(\rho_{ABC}) - \mathcal{S}_2(\rho_B)$$
$$= \frac{1}{2}\ln\left(\frac{\det\boldsymbol{\gamma}_{AB}\det\boldsymbol{\gamma}_{BC}}{\det\boldsymbol{\gamma}_{ABC}\det\boldsymbol{\gamma}_B}\right) \geq 0. \quad (10)$$

Proof.—The result follows by applying a particular norm compression inequality to the CM $\boldsymbol{\gamma}_{ABC}$. Given a positive Hermitian matrix $\boldsymbol{A} \in \mathbb{M}_m$, and given any two index sets $\alpha, \beta \subseteq N = \{1, \ldots, m\}$, the Hadamard-Fisher inequality [49] states that $\det\boldsymbol{A}_{\alpha\cup\beta}\det\boldsymbol{A}_{\alpha\cap\beta} \leq \det\boldsymbol{A}_\alpha\det\boldsymbol{A}_\beta$. Recalling that any CM $\boldsymbol{\gamma}_{ABC}$ is a positive real symmetric matrix [50], the claim follows upon identifying α with the indices of modes AB and β with the indices of modes BC. ∎

Beyond its apparent simplicity, Theorem 1 has profound consequences. It yields that the core of quantum information theory can be consistently reformulated, within the Gaussian scenario [24,25], using the simpler and physically natural Rényi-2 entropy as an alternative to the von Neumann one. In the rest of this Letter we will focus on defining GR2 quantifiers of entanglement and other correlations for Gaussian states.

Gaussian Rényi-2 measures of correlations.—The GR2 entanglement $\mathcal{E}_2$ can be defined as follows. Given an n-mode bipartite Gaussian state ρ_{AB} with CM $\boldsymbol{\gamma}_{AB}$,

$$\mathcal{E}_2(\rho_{A:B}) \doteq \inf_{\left\{\boldsymbol{\sigma}_{AB}: \substack{\pm i\boldsymbol{\omega}^{\oplus n} \leq \boldsymbol{\sigma}_{AB} \leq \boldsymbol{\gamma}_{AB}, \\ \det\boldsymbol{\sigma}_{AB}=1}\right\}} \frac{1}{2}\ln(\det\boldsymbol{\sigma}_A). \quad (11)$$

For a pure Gaussian state $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$, the minimum is saturated by $\boldsymbol{\sigma}_{AB} = \boldsymbol{\gamma}_{AB}$, so that $\mathcal{E}_2(\psi_{A:B}) = \mathcal{S}_2(\rho_A) = \frac{1}{2}\ln(\det\boldsymbol{\gamma}_A)$, where $\boldsymbol{\gamma}_A$ is the reduced CM of subsystem A. For a generally mixed state, Eq. (11)—where the minimization is over pure n-mode Gaussian states with CM $\boldsymbol{\sigma}_{AB}$ smaller than $\boldsymbol{\gamma}_{AB}$—amounts to taking the Gaussian convex roof of the pure-state Rényi-2 entropy of entanglement, according to the formalism of Refs. [21,32]. Closed formulae for $\mathcal{E}_2$ can be obtained for special classes of two-mode Gaussian states [33,51,52] exploiting the same procedure adopted for the Gaussian

entanglement of formation [32,48]. Like the latter measure (and all Gaussian convex-roof entanglement measures), it follows from the results of Ref. [32] that the GR2 entanglement is in general monotonically nonincreasing under Gaussian local operations and classical communication, and is additive for two-mode symmetric Gaussian states.

We can also introduce a 'GR2 measure of one-way classical correlations in the spirit of Henderson and Vedral [35]. We define $\mathcal{J}_2(\rho_{A|B})$ as the maximum decrease in the Rényi-2 entropy of subsystem A, given a Gaussian measurement has been performed on subsystem B, where the maximization is over all Gaussian measurements—i.e., those that map Gaussian states into Gaussian states [37,38]. Any such measurement on, say, the n_B-mode subsystem $B = (B_1 \ldots B_{n_B})$, is described by a positive operator valued measure [53] of the form $\Pi_B(\boldsymbol{\eta}) = \pi^{-n_B}[\prod_{j=1}^{n_B}\hat{W}_{B_j}(\eta_j)]\Lambda_B^\Pi[\prod_{j=1}^{n_B}\hat{W}_{B_j}^\dagger(\eta_j)]$ where $\hat{W}_B(\eta_j) = \exp(\eta_j\hat{b}_j^\dagger - \eta_j^*\hat{b}_j)$ is the Weyl operator, $\hat{b}_j = (\hat{q}_{B_j} + i\hat{p}_{B_j})/\sqrt{2}$, $\pi^{-n_B}\int\Pi_B(\boldsymbol{\eta})d^{2n_B}\boldsymbol{\eta} = \mathbb{1}$, and Λ_B^Π is the density matrix of a (generally mixed) n_B-mode Gaussian state with CM $\boldsymbol{\Gamma}_B^\Pi$ which denotes the seed of the measurement. The conditional state $\rho_{A|\boldsymbol{\eta}}$ of subsystem A after the measurement $\Pi_B(\boldsymbol{\eta})$ has been performed on B has a CM $\tilde{\boldsymbol{\gamma}}_A^\Pi$ independent of the outcome $\boldsymbol{\eta}$ and given by the Schur complement [54]

$$\tilde{\boldsymbol{\gamma}}_A^\Pi = \boldsymbol{\gamma}_A - \mathbf{s}_{AB}(\boldsymbol{\gamma}_B + \boldsymbol{\Gamma}_B^\Pi)^{-1}\mathbf{s}_{AB}^\top, \quad (12)$$

where the original bipartite CM $\boldsymbol{\gamma}_{AB}$ of the n-mode state ρ_{AB} has been written in block form as in (8). We have then

$$\mathcal{J}_2(\rho_{A|B}) \doteq \sup_{\boldsymbol{\Gamma}_B^\Pi}\frac{1}{2}\ln\left(\frac{\det\boldsymbol{\gamma}_A}{\det\tilde{\boldsymbol{\gamma}}_A^\Pi}\right). \quad (13)$$

The one-way classical correlations $\mathcal{J}_2(\rho_{B|A})$, with measurements on A, can be defined accordingly by swapping $A \leftrightarrow B$.

We can now define a Gaussian measure of quantumness of correlations based on Rényi-2 entropy. Following the seminal study by Ollivier and Zurek [34], and the recent analysis of Gaussian quantum discord using von Neumann entropy [37,38], we define the 'GR2 discord' as the difference between mutual information (9) and classical correlations (13),

$$\mathcal{D}_2(\rho_{A|B}) \doteq I_2(\rho_{A:B}) - \mathcal{J}_2(\rho_{A|B})$$
$$= \inf_{\boldsymbol{\Gamma}_B^\Pi}\frac{1}{2}\ln\left(\frac{\det\boldsymbol{\gamma}_B\det\tilde{\boldsymbol{\gamma}}_A^\Pi}{\det\boldsymbol{\gamma}_{AB}}\right). \quad (14)$$

For the case of A and B being single modes, that is, ρ_{AB} being a general two-mode Gaussian state, closed formulae can be obtained for Eqs. (13) and (14) thanks to the results of Ref. [37]. We report them in the Supplemental Material [48] for completeness. We remark that Theorem 1 is crucial to guarantee the non-negativity and the faithfulness of the GR2 discord [55–57]. Notice also that

$\frac{1}{2} \mathcal{I}_2(\rho_{A:B}) = \mathcal{J}_2(\rho_{A|B}) = \mathcal{D}_2(\rho_{A|B}) = \mathcal{S}_2(\rho_A)$ for pure bipartite Gaussian states ρ_{AB}.

A trade-off relation between the entanglement $\mathcal{E}_2$ and the classical correlations $\mathcal{J}_2$ can be written for arbitrary tripartite pure Gaussian states ρ_{ABC} [37], following Koashi and Winter [59]. One can essentially exploit the fact that all possible Gaussian positive operator valued measures on B induce all possible Gaussian pure-state decompositions of the subsystem AC (see also Ref. [16]), which implies

$$\mathcal{S}_2(\rho_A) = \mathcal{J}_2(\rho_{A|B}) + \mathcal{E}_2(\rho_{A:C}). \tag{15}$$

This relation can be manipulated to express the "conservation" of different types of correlations in a generic pure tripartite Gaussian state ρ_{ABC}, along the lines of Ref. [60].

We now look at monogamy properties of the GR2 measures. For an entanglement monotone E and an n-partite state $\rho_{A_1 A_2 \dots A_n}$, the monogamy relation (choosing party A_1 as the focus), which constrains the distribution of bipartite entanglement among different splits, can be written as Ref. [15] $E(\rho_{A_1 : A_2 \dots A_n}) - \sum_{j=2}^{n} E(\rho_{A_1 : A_j}) \geq 0$. The Rényi-2 entanglement measure [13,17], as well as the tangle (squared concurrence) [15,16], satisfy this inequality for general n-qubit states. A Gaussian version of the tangle (based on squared negativity) has been defined that obeys the inequality for all n-mode Gaussian states [40]. We now show that $\mathcal{E}_2$ does too.

Theorem 2.—The GR2 entanglement defined in Eq. (11) is monogamous for all n-mode Gaussian states $\rho_{A_1 A_2 \dots A_n}$,

$$\mathcal{E}_2(\rho_{A_1 : A_2 \dots A_n}) - \sum_{j=2}^{n} \mathcal{E}_2(\rho_{A_1 : A_j}) \geq 0, \tag{16}$$

where each A_j comprises one mode only.

Proof.—The structure of the proof follows closely the one for the tangle of n-qubit systems [16]. It suffices to prove the inequality for tripartite Gaussian states $\rho_{A_1 A_2 A_3'}$ where A_3' comprises $n - 2$ modes (with n arbitrary), as iterative applications to $\mathcal{E}_2(\rho_{A_1 : A_3'})$ would then imply Eq. (18). It is further enough to prove the inequality on pure states, as it would then extend to mixed ones by convexity [15,16,39,40]. Exploiting the phase-space Schmidt decomposition [21,43,61], when $\rho_{A_1 A_2 A_3'}$ is pure, the state of subsystem A_3' is locally equivalent to a 2-mode state, tensored by n-4 irrelevant vacuum modes. The problem reduces to proving that $\mathcal{E}_2(\rho_{A:BC}) \geq \mathcal{E}_2(\rho_{A:B}) + \mathcal{E}_2(\rho_{A:C})$ for an arbitrary pure Gaussian state ρ_{ABC}, where A and B are single modes, while C groups two modes. Noting that $\mathcal{E}_2(\rho_{A:BC}) = \mathcal{S}_2(\rho_A)$, and exploiting Eq. (15), we see that Eq. (16) is verified if one establishes that

$$\mathcal{J}_2(\rho_{A|B}) \geq \mathcal{E}_2(\rho_{A:B}) \tag{17}$$

holds for all (mixed) two-mode Gaussian states ρ_{AB}. The inequality (17) is proven in the Supplemental Material [48], which concludes the proof of the Theorem. ∎

Let us analyze in detail the case of $\rho_{A_1 A_2 A_3}$ being a pure 3-mode Gaussian state, whose CM is characterized up to

local unitaries by three parameters (local symplectic invariants) $a_j \geq 1$, with $\mathcal{S}_2(\rho_{A_j}) = \ln a_j$ ($j = 1, 2, 3$) [48,52]. We define the residual entanglement emerging from the monogamy inequality as $\mathcal{E}_2(\rho_{A_1 : A_2 : A_3}) = \mathcal{E}_2(\rho_{A_1 : A_2 A_3}) - \mathcal{E}_2(\rho_{A_1 : A_2}) - \mathcal{E}_2(\rho_{A_1 : A_3})$. This quantity, which can be calculated exactly [39,52], depends in general on the focus mode (say A_1) chosen for the decomposition of the bipartite entanglements. Remarkably, we find that for all pure three-mode Gaussian states which are fully inseparable and display entanglement in all global and reduced bipartitions ($\mathcal{E}_2(\rho_{A_i : A_j}) > 0$, $\mathcal{E}_2(\rho_{A_i : A_j A_k}) > 0$, $\forall~i \neq j \neq k$), the residual GR2 entanglement $\mathcal{E}_2(\rho_{A_1 : A_2 : A_3})$ is invariant under mode permutations, thus representing (to the best of our knowledge) the first and only known intrinsically *bona fide* measure of genuine tripartite Gaussian entanglement [62]. We report its explicit formula here, while a derivation is provided in Ref. [48],

$$\mathcal{E}_2(\rho_{A_1 : A_2 : A_3}) = \ln(8 a_1 a_2 a_3) - \ln\left[-1 - \sqrt{\delta} \right.$$
$$\left. + \sum_{i=1}^{3} (2 a_i^2 - a_i^4) + \sum_{i \neq j = 1}^{3} (a_i^2 a_j^2) \right], \tag{18}$$

with $\delta = \prod_{\mu, \nu = 0}^{1} \{ [a_1 + (-1)^\mu a_2 + (-1)^\nu a_3]^2 - 1 \}$. The formula (18) holds when $|a_i - a_j| + 1 < a_k < (a_i^2 + a_j^2 - 1)^{1/2}$.

We finally remark that, importantly, the GR2 discord Eq. (14) also turns out to be a monogamous measure of quantum correlations for arbitrary pure three-mode Gaussian states $\rho_{A_1 A_2 A_3}$ (unlike the von Neumann entropy-based discord [37,38,41]). By using Eq. (15), one finds (see also Refs. [59,60]) $\mathcal{D}_2(\rho_{A_1 : A_2 : A_3}) \doteq \mathcal{D}_2(\rho_{A_2 | A_2 A_3}) - \mathcal{D}_2(\rho_{A_1 | A_2}) - \mathcal{D}_2(\rho_{A_1 | A_3}) = \mathcal{E}_2(\rho_{A_1 : A_2 : A_3})$. In other words, the residual tripartite discord equates the residual tripartite entanglement for pure $\rho_{A_1 A_2 A_3}$. This extends to the multipartite case the equivalence between entanglement and general quantum correlations valid for pure bipartite states [34,64,65], and places $\mathcal{D}_2$ as the only known measure of quantumness beyond entanglement in continuous variable systems that fulfills monogamy [66].

Conclusions.—In this Letter we planted the seeds for a full Gaussian quantum information theory [25] using the Rényi-2 entropy $\mathcal{S}_2$. This is possible thanks to the fact, proven in Theorem 1, that such an entropy satisfies the strong subadditivity inequality for arbitrary Gaussian states of quantum harmonic systems. We employed $\mathcal{S}_2$ to define valid measures of entanglement, total, classical, and quantum correlations, highlighting their properties. The Rényi-2 mutual information is intimately related to Wigner distribution sampling by homodyne detections in phase space. The residual Rényi-2 entanglement measure allows for a quantification of genuine tripartite entanglement in three-mode fully inseparable pure Gaussian states, which is invariant under mode permutations. We argue that

the measures defined in this Letter should be adopted as privileged tools to address the quantification of relevant correlations in Gaussian states. A very recent application to relativistic quantum information has been reported [68].

By conception, this work has been biased towards Gaussian states and operations. However, the approach pursued here can be extended to arbitrary, even non-Gaussian n-mode states ρ of continuous variable systems, provided one chooses the Wehrl entropy [6] to quantify the informational content of ρ. Such an entropy is operationally associated to phase-space sampling via heterodyne detections [45] as it corresponds to the continuous Boltzmann-Gibbs-Shannon entropy of the Husimi Q distribution of ρ, $Q(\boldsymbol{\alpha}) = \pi^{-n}\langle\boldsymbol{\alpha}|\rho|\boldsymbol{\alpha}\rangle$, which is a valid (non-negative) probability distribution for *all* quantum states [1]. One can then adopt a distance measure between any two ρ_1 and ρ_2 in terms of the relative Shannon entropy between their respective Q distributions, define ensuing correlation measures, and so on. Such a formalism could also accommodate a measure of non-Gaussianity of quantum states [69], and one might then naturally compare Gaussian with non-Gaussian operations for the realization of specific tasks [27] such as extracting classical correlations [70,71], maximizing information-disturbance trade off [72] and performing optimal cloning of coherent states [73]. This will be the subject of further study.

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