Berry Curvature, Triangle Anomalies, and the Chiral Magnetic Effect in Fermi Liquids

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In a three-dimensional Fermi liquid, quasiparticles near the Fermi surface may possess a Berry curvature. We show that if the Berry curvature has a nonvanishing flux through the Fermi surface, the particle number associated with this Fermi surface has a triangle anomaly in external electromagnetic fields. We show how Landau's Fermi liquid theory should be modified to take into account the Berry curvature. We show that the "chiral magnetic effect" also emerges from the Berry curvature flux.

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Introduction.—Recently there has been a lot of interest in the effect of the Berry phase and Berry curvature on the physics of the electron Fermi liquid. The standard theory of Fermi liquids, developed by Landau [1], assumes that the low-energy degrees of freedom in a Fermi liquid are the fermion quasiparticles, whose distribution function in phase space satisfies a kinetic equation. In many cases the semiclassical motion of a wave packet of electrons in a crystal should include an extra term due to the Berry phase, expressible in terms of the electronic Bloch wave functions [2]. Such a term should alter the standard kinetics of Fermi liquids (see also below); including this term leads to an interpretation of the anomalous Hall conductivity in terms of Fermi surface properties [3].

In this Letter we show the connection between the Berry curvature on the Fermi surface and triangle anomalies. Let us first notice that the total flux of Berry curvature through a given Fermi surface does not need to vanish, but can be a multiple of the flux quantum. One possible case is doped Weyl semimetals [4–6], but the discussion does not depend on the origin of the Berry curvature flux. We will show that if there are k quanta of Berry curvature flux through a given Fermi surface, then the number of particles associated with this Fermi surface (which is proportional to its volume) is not conserved in the presence of the external electromagnetic field,

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{j} = \frac{k}{4\pi^2}\mathbf{E}\cdot\mathbf{B}. \qquad (1)$$

Charge conservation is ensured by the vanishing of the sum of k's of all Fermi surfaces.

Equation (1) is exactly the equation of triangle anomalies in relativistic quantum field theory [7,8]. It is therefore expected to hold for a Fermi gas of relativistic fermions at finite density. Indeed, the Berry curvature of a relativistic fermion has the form of the field of a magnetic monopole in momentum space, and $k = \pm 1$ for right- (left-)handed fermions. The statement (1) goes further by tying anomalies with Fermi surface properties only. In this way, we demonstrate that axial anomalies are properties of Fermi liquids with Berry curvature flux, even when the original particles interact strongly. The only assumption is that low-energy degrees of freedom are fermions that are described by Landau's Fermi liquid theory; the extension to, e.g., the superfluid A phase of ^{3}He [9] is deferred to a future question, where Nambu-Goldstone bosons associated with the spontaneous breaking of U(1) particle number must also be taken into account. (In such a Weyl superfluid, the relation between the topology of a Fermi surface and anomalies was studied theoretically in a different way and verified experimentally; see Ref. [9] and references therein.)

As is evident from our arguments, triangle anomalies in a Fermi liquid have a "kinematic" origin, independent of the details of the Hamiltonian. Namely, we will show that Berry curvature modifies the commutation relation of the particle number density operator, and that this commutator is related to the anomalous Hall effect and the triangle anomalies for the fermion numbers near one Fermi surface.

Hamiltonian formulation of Landau's Fermi liquid theory.—The fundamental equation of Landau's Fermi liquid theory is a kinetic equation governing the time evolution of the occupation number of quasiparticles $n_\mathbf{p}(\mathbf{x})$,

$$\frac{\partial n_\mathbf{p}(\mathbf{x})}{\partial t} + \frac{\partial \epsilon_\mathbf{p}}{\partial \mathbf{p}}\cdot\frac{\partial n_\mathbf{p}}{\partial \mathbf{x}} - \frac{\partial \epsilon_\mathbf{p}}{\partial \mathbf{x}}\cdot\frac{\partial n_\mathbf{p}}{\partial \mathbf{p}} = 0, \qquad (2)$$

where $\epsilon_\mathbf{p} = \epsilon^0_\mathbf{p} + \delta\epsilon_\mathbf{p}$, $\epsilon^0_\mathbf{p}$ is the energy of a single quasiparticle excitation with energy $\mathbf{p}$, and $\delta\epsilon_\mathbf{p}$ is the modification of its energy due to interactions with other quasiparticles,

$$\delta\epsilon_\mathbf{p} = \int \frac{d\mathbf{q}}{(2\pi)^3} f(\mathbf{p},\mathbf{q})\delta n_\mathbf{q}(\mathbf{y}), \qquad (3)$$

$\delta n_\mathbf{q}(\mathbf{y}) = n_\mathbf{q}(\mathbf{y}) - n^0_\mathbf{q}$ is the deviation from the ground state distribution function and $f(\mathbf{p},\mathbf{q})$ are Landau's parameter. Above we have neglected the collision term.

For the purpose of generalizing Landau's Fermi liquid theory to systems with Berry curvature, we reformulate the kinetic equation as the evolution equation of a Hamiltonian system. In this formulation, the kinetic equation has the form

$$\partial_t n_\mathbf{p}(\mathbf{x}) = i[H, n_\mathbf{p}(\mathbf{x})], \qquad (4)$$

where the Hamiltonian H is the conserved energy,

$$H = \int \frac{d\mathbf{p}d\mathbf{x}}{(2\pi)^3}\epsilon^0_{\mathbf{p}}\delta n_{\mathbf{p}} + \frac{1}{2}\int \frac{d\mathbf{p}d\mathbf{q}d\mathbf{x}}{(2\pi)^6} f(\mathbf{p},\mathbf{q})\delta n_{\mathbf{p}}\delta n_{\mathbf{q}}, \tag{5}$$

and the commutator is postulated as

$$[n_{\mathbf{p}}(\mathbf{x}), n_{\mathbf{q}}(\mathbf{y})] = -i(2\pi)^3 \frac{\partial}{\partial \mathbf{p}}\delta(\mathbf{p}-\mathbf{q})\cdot\frac{\partial}{\partial \mathbf{x}}\delta(\mathbf{x}-\mathbf{y}) \times [n_{\mathbf{p}}(\mathbf{y}) - n_{\mathbf{q}}(\mathbf{x})]. \tag{6}$$

It is straightforward to verify that Eqs. (4)–(6) imply Eq. (2).

The commutation relation (6) is remarkable in the following respect: assume we have two operators, $\hat{A}$ and $\hat{B}$, linear in occupation numbers,

$$\hat{A} = \int \frac{d\mathbf{p}d\mathbf{x}}{(2\pi)^3} A(\mathbf{p},\mathbf{x}) n_{\mathbf{p}}(\mathbf{x}), \qquad \hat{B} = \int \frac{d\mathbf{p}d\mathbf{x}}{(2\pi)^3} B(\mathbf{p},\mathbf{x}) n_{\mathbf{p}}(\mathbf{x}), \tag{7}$$

then its commutator will be

$$[\hat{A},\hat{B}] = -i\int \frac{d\mathbf{p}d\mathbf{x}}{(2\pi)^3}\{A,B\} n_{\mathbf{p}}(\mathbf{x}), \tag{8}$$

where $\{A, B\}$ is the classical Poisson bracket

$$\{A,B\}(\mathbf{p},\mathbf{x}) = \frac{\partial A}{\partial \mathbf{p}}\cdot\frac{\partial B}{\partial \mathbf{x}} - \frac{\partial A}{\partial \mathbf{x}}\cdot\frac{\partial B}{\partial \mathbf{p}}. \tag{9}$$

The presence of the Berry curvature, as we shall see, changes the classical Poisson bracket and leads to a modification of Landau's Fermi liquid theory.

Berry curvature and Poisson brackets.—Before tackling the many-body physics of Fermi liquids, let us consider a single quasiparticle in a theory with Berry curvature of the Fermi surface. Such a quasiparticle is described by the action [10,11]

$$S = \int dt[p^i\dot{x}^i + A_i(x)\dot{x}^i - \mathcal{A}_i(p)\dot{p}^i - H(p,x)], \tag{10}$$

where $H(p, x)$ is the Hamiltonian whose form is not important for us right now, A_i is the electromagnetic vector potential, and $\mathcal{A}_i(p)$ is a fictitious vector potential in momentum space. Combining p and x into one set of variables ξ^a, $a = 1, \ldots, 6$, the action can be written as

$$S = \int dt[-\omega_a(\xi)\dot{\xi}^a - H(\xi)]. \tag{11}$$

The equations of motion that follow from this action are

$$\omega_{ab}\dot{\xi}^b = -\partial_a H, \tag{12}$$

where $\omega_{ab} = \partial_a\omega_b - \partial_b\omega_a$ and $\partial_a \equiv \partial/\partial\xi^a$. We can reinterpret this equation as

$$\dot{\xi}^a = \{H, \xi^a\} = -\{\xi^a, \xi^b\}\partial_b H, \tag{13}$$

where the Poisson bracket is defined as

$$\{\xi^a, \xi^b\} = (\omega^{-1})^{ab} \equiv \omega^{ab}, \tag{14}$$

where ω^{-1} is the matrix inverse of ω_{ab}. For the action (10), the Poisson brackets are [11]

$$\{p_i, p_j\} = -\frac{\epsilon_{ijk}B_k}{1+\mathbf{B}\cdot\mathbf{\Omega}}, \tag{15a}$$

$$\{x_i, x_j\} = \frac{\epsilon_{ijk}\Omega_k}{1+\mathbf{B}\cdot\mathbf{\Omega}}, \tag{15b}$$

$$\{p_i, x_j\} = \frac{\delta_{ij} + \Omega_i B_j}{1+\mathbf{B}\cdot\mathbf{\Omega}}, \tag{15c}$$

where $B_i = \epsilon_{ijk}\partial A_k/\partial x_j$, $\Omega_i = \epsilon_{ijk}\partial\mathcal{A}_k/\partial p_j$.

The invariant phase space is (here $\omega \equiv \det\omega_{ab}$) [10]

$$d\Gamma = \sqrt{\omega}d\xi = (1+\mathbf{B}\cdot\mathbf{\Omega})\frac{d\mathbf{p}d\mathbf{x}}{(2\pi)^3}. \tag{16}$$

It is now clear how to incorporate Berry curvature into Landau's Fermi liquid theory. One makes a phase-space modification to the Hamiltonian (5), keeps the evolution Eq. (4) unchanged, but alters the commutator of $n_{\mathbf{p}}(\mathbf{x})$ to be consistent with Eqs. (15). We shall now work out this commutator.

Let us assume that there are two operators $\hat{A}$ and $\hat{B}$ defined as

$$\hat{A} = \int d\xi\sqrt{\omega}A(\xi)n(\xi), \quad \hat{B} = \int d\xi\sqrt{\omega}B(\xi)n(\xi). \tag{17}$$

Then it seems natural to define the commutator between $n(\xi)$ so that

$$[\hat{A},\hat{B}] = -i\int d\xi\sqrt{\omega}\omega^{ab}\partial_a A\partial_b B n(\xi). \tag{18}$$

This form, however, is deficient in one respect: it makes use of $n(\xi)$ in the whole Fermi volume, while we expect the physics to be concentrated near the Fermi surface. We shall therefore postulate another form for the commutator,

$$[\hat{A},\hat{B}] = -\frac{i}{2}\int d\xi\sqrt{\omega}\omega^{ab}(A\partial_a B - B\partial_a A)\partial_b n(\xi). \tag{19}$$

If we integrate by part in this equation, using $\partial_b(\sqrt{\omega}\omega^{ab}) = 0$ (a consequence of the Bianchi identity), we bring Eq. (19) into the form of (18). However, now the commutator depends only on the physics near the Fermi surface. Moreover, we may have problems defining the integral in Eq. (18) when the Berry curvature is singular inside the Fermi volume (as in the case when the Berry curvature flux is nonzero), while Eq. (19) is completely well defined in this case. We will take Eq. (19) as the equation defining the commutators of the occupation number operator.

It is possible to write down explicitly the commutator $[n_{\mathbf{p}}(\mathbf{x}), n_{\mathbf{q}}(\mathbf{y})]$. We shall not do it here. Instead, we shall notice that if A and B are not linear in $n(\xi)$, but are instead general functionals of $n(\xi)$, then the commutator between them can still be computed explicitly,

$$[\hat{A}, \hat{B}] = -\frac{i}{2}\int d\xi \sqrt{\omega}\,\omega^{ab}\left(\frac{\delta \hat{A}}{\delta n(\xi)}\partial_a \frac{\delta \hat{B}}{\delta n(\xi)} - \frac{\delta \hat{B}}{\delta n(\xi)}\partial_a \frac{\delta \hat{A}}{\delta n(\xi)}\right)\partial_b n(\xi). \quad (20)$$

Equation (20) is particularly useful when $\hat{A}$ is the Hamiltonian, for which we know that $\delta H/\delta n_{\mathbf{p}}(\mathbf{x}) = \epsilon_{\mathbf{p}}(\mathbf{x})$.

Commutator of density operator.—We now show that the Berry curvature leads to an anomalous term in the equal-time commutator of the density operator $n(\mathbf{x})$ at two points. Moreover, if the Berry curvature has a nonzero magnetic flux through the Fermi sphere, then the commutator has a contribution from the external magnetic field,

$$[n(\mathbf{x}), n(\mathbf{y})] = -i\left(\nabla \times \boldsymbol{\sigma} + \frac{k}{4\pi^2}\mathbf{B}\right)\cdot \nabla \delta(\mathbf{x}-\mathbf{y}), \quad (21)$$

where $\boldsymbol{\sigma}$ is defined as

$$\sigma_i(\mathbf{x}) = -\int \frac{d\mathbf{p}}{(2\pi)^3} p_i \Omega_k \frac{\partial n_{\mathbf{p}}(\mathbf{x})}{\partial p_k}, \quad (22)$$

and k is the monopole charge inside the Fermi surface,

$$k = \frac{1}{2\pi}\int d\mathbf{S}\cdot \boldsymbol{\Omega}. \quad (23)$$

We note that both $\boldsymbol{\sigma}$ and k involve only the physics near the Fermi surface.

To derive Eq. (21), first we write the density operator as

$$n(\mathbf{y}) = \int \frac{d\mathbf{p}}{(2\pi)^3}(1 + \mathbf{B}\cdot\boldsymbol{\Omega})n_{\mathbf{p}}(\mathbf{y}) = \int d\Gamma\, \delta(\mathbf{x}-\mathbf{y}) n_{\mathbf{p}}(\mathbf{x}). \quad (24)$$

The commutator of the density operator at two different points is, according to Eq. (19),

$$[n(\mathbf{y}), n(\mathbf{z})] = -\frac{i}{2}\int d\Gamma\, \delta(\mathbf{x}-\mathbf{y})\partial_i \delta(\mathbf{x}-\mathbf{z})\left[\{x_i, x_j\}\frac{\partial n_{\mathbf{p}}}{\partial x_j} + \{x_i, p_j\}\frac{\partial n_{\mathbf{p}}}{\partial p_j}\right] - (\mathbf{y} \leftrightarrow \mathbf{z}). \quad (25)$$

The $\{x_i, x_j\}$ term in the commutator is reduced to

$$-i\partial_i \delta(\mathbf{y}-\mathbf{z})\int \frac{d\mathbf{p}}{(2\pi)^3}\epsilon_{ijk}\Omega_k \frac{\partial n_{\mathbf{p}}}{\partial x_j} = -i(\nabla \times \boldsymbol{\sigma})\cdot \nabla \delta(\mathbf{y}-\mathbf{z}), \quad (26)$$

where $\boldsymbol{\sigma}$ is defined in Eq. (22). The $\{x_i, p_j\}$ term in the commutator can be rewritten as

$$iB_i \partial_i \delta(\mathbf{y}-\mathbf{z})\int \frac{d\mathbf{p}}{(2\pi)^3}\Omega_j \frac{\partial n_{\mathbf{p}}}{\partial p_j}, \quad (27)$$

and, by integration by part, taking into account $\partial_i \Omega_i = 0$ around the Fermi surface, $n_{\mathbf{p}} = 1$ deep inside the Fermi surface and $n_{\mathbf{p}} = 0$ far outside the Fermi sphere, it becomes

$$-i\frac{k}{4\pi^2}\mathbf{B}\cdot \nabla \delta(\mathbf{y}-\mathbf{z}). \quad (28)$$

Combining two contributions, we find Eq. (21).

From density-density commutator to anomalous non-conservation of current.—The connection between the anomalous density-density commutator [the term proportional to $\mathbf{B}$ in Eq. (21)] and triangle anomalies is known in the context of relativistic quantum field theory [12,13]. Here we derive this connection using the Hamiltonian formalism and show how the anomalous Hall current and the triangle anomaly can be traced to the two contributions to the density-density commutator.

Let us first assume that our system is in a static magnetic field, but the electric field is turned off. In this case, the system is described by the Hamiltonian (5), and by commuting the Hamiltonian with the particle number operator $n(\mathbf{x})$, the continuity equation can be derived,

$$\dot{n} = i[H, n] = -\nabla \cdot \mathbf{j}, \quad (29)$$

where the particle number current $\mathbf{j}$ is

$$\mathbf{j} = \int \frac{d\mathbf{p}}{(2\pi)^3}\left[-\epsilon_{\mathbf{p}}\frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} - \left(\boldsymbol{\Omega}\cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}}\right)\epsilon_{\mathbf{p}}\mathbf{B} - \epsilon_{\mathbf{p}}\boldsymbol{\Omega}\times \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{x}}\right]. \quad (30)$$

Note that by integration by part, the first term in the brackets in the right-hand side of Eq. (30) can be written in the familiar form $n_{\mathbf{p}}\mathbf{v}$, where $\mathbf{v} = \partial \epsilon_{\mathbf{p}}/\partial \mathbf{p}$. This would be the only term in the current in the absence of Berry curvature.

Now we turn on a static electric field by putting the system in an external scalar potential $\phi(\mathbf{x})$, $\mathbf{E} = -\nabla \phi$. The Hamiltonian is now

$$H' = H + \int d\mathbf{x}\, \phi(\mathbf{x}) n(\mathbf{x}). \quad (31)$$

The added term does not commute with n and changes the time evolution of the latter,

$$\partial_t n(\mathbf{x}) = i[H', n(\mathbf{x})] = -\nabla \cdot \mathbf{j} - \left(\nabla \times \boldsymbol{\sigma} + \frac{k}{4\pi^2}\mathbf{B}\right)\cdot \nabla \phi(\mathbf{x}). \quad (32)$$

This equation can be rewritten as

$$\partial_t n + \nabla \cdot \mathbf{j}' = \frac{k}{4\pi^2}\mathbf{E}\cdot \mathbf{B}, \quad (33)$$

where

$$\mathbf{j}' = \mathbf{j} + \mathbf{E}\times \boldsymbol{\sigma}. \quad (34)$$

The second term in Eq. (34) is the usual anomalous Hall current. On the other hand, Eq. (33) implies that the particle number around the Fermi surface is not conserved when both electric and magnetic fields are turned on. This is the effect of triangle anomalies in quantum field theory. For example, relativistic right-handed free fermions have

$k = 1$, and left-handed free fermions have $k = -1$. Here we show that this effect depends only on the monopole charge of the Berry curvature on the Fermi surface and is not modified by interactions. Since the total charge is conserved, all different contributions to the current nonconservation should sum up to zero.

Chiral magnetic effect.—Let us compute the current, given by Eq. (30), in the thermal equilibrium state, where quasiparticles have a Fermi-Dirac distribution function,

$$n_{\mathbf{p}} = f(x) = \frac{1}{e^x + 1}, \qquad x = \frac{\epsilon_{\mathbf{p}} - \mu}{T}. \tag{35}$$

There are three contributions to $\mathbf{j}$ corresponding to three terms in the brackets in the right-hand side of Eq. (30). The third term involves spatial derivatives and vanishes in the ground state. We now show that the first term also vanishes identically. For this end it is useful to introduce the function $g(x) = \int_{-\infty}^{x} dy y f'(y)$, for which $g(-\infty) = g(+\infty) = 0$. Then

$$-\int \frac{d\mathbf{p}}{(2\pi)^3} \epsilon_{\mathbf{p}} \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} = -\int \frac{d\mathbf{p}}{(2\pi)^3} \left[\mu \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} + T \frac{\partial}{\partial \mathbf{p}} g(x) \right] = 0. \tag{36}$$

Similarly, the second contribution can be written as

$$-\mathbf{B} \int \frac{d\mathbf{p}}{(2\pi)^3} \cdot \left[\mu \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} + T \frac{\partial}{\partial \mathbf{p}} g(x) \right], \tag{37}$$

and the integrals can be evaluated as in Eq. (27). As the result, we find

$$\mathbf{j} = \frac{k}{4\pi^2} \mu \mathbf{B}. \tag{38}$$

Let us assume for definiteness that there are two Fermi surfaces with $k = 1$ and $k = -1$. If the two Fermi surfaces have equal chemical potential, then the total current is equal to 0. However, if the chemical potentials are unequal (which can be achieved by turning on an $\mathbf{E} \cdot \mathbf{B}$ for a finite time), then there will be a current equal to $(\mu_+ - \mu_-)\mathbf{B}/4\pi^2$ in this quasiequilibrium state. This is the chiral magnetic effect [14,15].

Conclusion.—The calculation above ties the anomalous current nonconservation to a property of the Fermi surface only (the Berry curvature) and hence can be applied to Fermi liquids, even when the interaction between original fermions is strong. This is done by using a kinetic equation for quasiparticles; a more microscopic derivation of this equation [16,17] is desirable.

It would be interesting to explore further physical consequences of Berry curvature on Landau Fermi liquid theory. Particularly interesting are the effects of Berry curvature on the collective modes and on the response of the Fermi liquids. It is also interesting to include the collision term into the kinetic equation and investigate the hydrodynamic regime. In relativistically invariant theories, the effects of triangle anomalies have been investigated, both within hydrodynamics and by using gauge-gravity duality [18–20], and there have been attempts to derive hydrodynamics from kinetic theory [21]. The kinetic approach allows us to go beyond the hydrodynamic regime and beyond systems with relativistic invariance. On the other hand, some interesting phenomena associated with anomalies in relativistic theories, like the Alfvén-type modes propagating along the direction of the magnetic field [22,23], may be directly investigated using the kinetic equation in the condensed-matter context.

Finally, the understanding obtained here should allow one to formulate the criteria of anomalies matching for dense states matter, i.e., quark matter phases with Fermi surfaces.

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