Device-Independent Bounds for Hardy's Experiment

Rafael Rabelo,[1] Law Yun Zhi,[2] and Valerio Scarani[1,2]

[1]Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543, Singapore
[2]Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117542, Singapore
(Received 18 July 2012; published 31 October 2012)

In this Letter, we compute an analogue of Tsirelson's bound for Hardy's test of nonlocality, that is, the maximum violation of locality constraints allowed by the quantum formalism, irrespective of the dimension of the system. The value is found to be the same as the one achievable already with two-qubit systems, and we show that only a very specific class of states can lead to such maximal value, thus highlighting Hardy's test as a device-independent self-test protocol for such states. By considering realistic constraints in Hardy's test, we also compute device-independent upper bounds on this violation and show that these bounds are saturated by two-qubit systems, thus showing that there is no advantage in using higher-dimensional systems in experimental implementations of such a test.

DOI: 10.1103/PhysRevLett.109.180401 PACS numbers: 03.65.Ud, 03.65.Wj, 03.67.Mn

Introduction.—The development of quantum information science is based on a recurrent pattern: nonclassical features of quantum physics, previously considered as mind-boggling and worth only of philosophical chat, are found to have an operational meaning and even to be potentially useful for applications. One of the discoveries that triggered this development is the prediction and observation of the violation of Bell inequalities [1]. This observation implies that correlations obtained by measuring separated quantum systems locally cannot be simulated classically without communication, a fact that is often referred to as *nonlocality.*

Within quantum information, nonlocality has undergone an interesting parable. For many years, it has been put aside as having fulfilled its role: the loathed local variables models having been disposed of forever, one could peacefully concentrate on entanglement theory. Only a few researchers kept on believing that this very intriguing observation could be useful for something in itself. The latter view was vindicated a few years ago, when it was noticed that nonlocality allows *device-independent* assessments: indeed, nonlocality is assessed only from the input-output statistics of the measurement, without reference to the degree of freedom that is being measured. This powerful type of assessment is sensitive to the existence of undesired side channels and will be ideal for certification of future quantum devices. So far, device-independent results are available for the security of quantum cryptography [2,3], the quality of sources [4,5] and measurement devices [6], and the amount of randomness that one can generate [7,8]. In this Letter, we study the possibility of device-independent assessment of one of the earliest proposals to check nonlocality: it used to be called *Hardy's paradox* but, in the spirit of quantum information, we would rather call it *Hardy's test* [9].

Hardy's test was originally stated by means of a particular experimental setup consisting of two overlapping Mach—Zehnder interferometers, one for electrons and one for positrons, arranged so that if the positron and the electron each take a particular path, they will meet and annihilate one another. A paradox arises under the assumption of local realism: in any classical local theory, a certain detection pattern must never occur, while quantum theory assigns to its occurrence a nonzero probability, hereafter referred to as *Hardy's probability.* It was soon realized that the argument could be extended to different states and measurements [10,11], and proved to hold for almost all entangled pure states of two qubits, with maximum Hardy's probability equal to

$$p_{\text{Hardy}} = (5\sqrt{5} - 11)/2 \approx 9\%. \tag{1}$$

Interestingly, though, the maximally entangled state of two qubits does not show nonlocality in Hardy's test.

Hardy's test has been the object of several theoretical generalizations [12–20] and has been implemented in experiments using photonic systems [21–25]. The latter, however, had to consider deviations from the original proposal, where the probabilities of a set of observations—hereafter, referred to as *constraint probabilities*—were assumed to be strictly equal to zero, an obviously unrealistic requirement. One way to overcome this problem is to consider a nonideal version of Hardy's test, and to compute local bounds on Hardy's probability in terms of the relaxed bounds on the constraint probabilities. The computed local bound, which a successful experiment must violate [26–28], turns out to be equivalent to the Clauser–Horne (CH) Bell inequality [29].

In this Letter, we provide three device-independent results on Hardy's test. First, we consider the original (or *ideal*) Hardy's test and prove that Eq. (1) is the maximum value of p_{Hardy} allowed by quantum physics, irrespective of the dimension; this is the analog of the Tsirelson bound [30]. A remarkable consequence of our derivation constitutes our second main result: any state that achieves Eq. (1)

in the ideal test is equivalent, up to local isometries, to the unique two-qubit state that achieves that violation. This is a case of *self-testing* [5,31,32]—a protocol for testing quantum systems and circuits that can lead to conclusive results even when the devices employed are not trusted—the first that detects a nonmaximally entangled state (see parallel work in Ref. [33]). Finally, our third result is a proof that, even for *nonideal* versions of Hardy's test, there is no practical advantage in using higher-dimensional systems.

Hardy's test.—Let us briefly summarize Hardy's test. Consider two parties, say, Alice and Bob, each of which is able to perform two possible measurements, $x = \{A_0, A_1\}$ and $y = \{B_0, B_1\}$, respectively, on its part of a shared physical system. Each measurement has two mutually exclusive outcomes, labeled by $a = \{\pm 1\}$, for the measurements of Alice, and $b = \{\pm 1\}$, for the ones of Bob (Fig. 1). The situation considered by Hardy assumes the three constraint probabilities

$$p(+, +|A_0, B_0) = 0, \tag{2a}$$
$$p(+, -|A_1, B_0) = 0, \tag{2b}$$
$$p(-, +|A_0, B_1) = 0. \tag{2c}$$

Suppose there are measurement devices and physical systems such that these three equations are fulfilled. If this setup can be described by a local realistic theory, then it follows that

$$p_{\text{Hardy}} \equiv p(+, +|A_1, B_1) = 0. \tag{3}$$

Hardy realized that, in quantum mechanics, there are measurements and a particular state of a two-qubit system such that the constraint probabilities are fulfilled while Hardy's probability is nonzero, leading to a so-called *paradox*. Extending the analysis to all possible measurements and states, Hardy later showed that the maximum value of p_{Hardy} for systems of two qubits is $(5\sqrt{5} - 11)/2$. A brute force calculation proved that this value cannot be exceeded using two three-dimensional systems [19]. Here, we prove that this value is device-independent, that is, it is optimal for bipartite quantum systems of any dimension.

Theorem 1.—The maximum value of Hardy's probability for quantum systems of arbitrary finite dimension is $p_{\text{Hardy}} = (5\sqrt{5} - 11)/2$, just as for qubits.

Proof.—In quantum mechanics, joint probabilities for the outcomes of measurements performed on spacelike separated parts of a quantum system are given by

FIG. 1 (color online). Schematic diagram for the Hardy's test scenario.

$$p(a, b|x, y) = \text{Tr}(\rho \Pi_{a|x} \otimes \Pi_{b|y}), \tag{4}$$

where ρ is the state of the system and $\Pi_{a|x}$, $\Pi_{b|y}$ are the measurement operators associated to outcomes a, b of measurements x, y, respectively. The latter operators are positive operator-value measure effects, in general; however, since we do not set any constraint on the dimension of the Hilbert space, Neumark's theorem allows us to consider only projective measurements, without loss of generality. The core of the proof exploits the following lemma, proven in Ref. [34]:

Lemma 1.—Given two Hermitian operators A_0 and A_1 with eigenvalues ± 1 acting on a Hilbert space $\mathcal{H}$, there is a decomposition of $\mathcal{H}$ as a direct sum of subspaces $\mathcal{H}^i$ of dimension $d \leq 2$ each, such that both A_0 and A_1 act within each $\mathcal{H}^i$, that is, they can be written as $A_0 = \oplus_i A_0^i$ and $A_1 = \oplus_i A_1^i$, where A_0^i and A_1^i act on $\mathcal{H}^i$.

Let then $A_0 = \Pi_{+|A_0} - \Pi_{-|A_0}$ and $A_1 = \Pi_{+|A_1} - \Pi_{-|A_1}$, where $\Pi_{a|x}$ are projection operators. It follows from Lemma 1 that $\Pi_{a|x} = \oplus_i \Pi^i_{a|x}$, where each $\Pi^i_{a|x}$ acts on $\mathcal{H}^i$, for all a and x; we also denote $\Pi^i = \Pi^i_{+1|x} + \Pi^i_{-1|x}$ the projector on $\mathcal{H}^i$. Needless to say, Lemma 1 is also valid on Bob's side; we use analog notations for Bob's operators. With these notations,

$$p(a, b|x, y) = \sum_{i,j} q_{ij} \text{Tr}(\rho_{ij} \Pi^i_{a|x} \otimes \Pi^j_{b|y}), \tag{5a}$$
$$\equiv \sum_{i,j} q_{ij} p_{ij}(a, b|x, y), \tag{5b}$$

where $q_{ij} = \text{Tr}(\rho \Pi^i \otimes \Pi^j)$ and $\rho_{ij} = (\Pi^i \otimes \Pi^j \rho \Pi^i \otimes \Pi^j)/q_{ij}$ is, at most, a two-qubit state. Since $q_{ij} \geq 0$ for all i, j and $\sum_{i,j} q_{ij} = 1$, the constraint probabilities [Eq. (2)] are satisfied for p if, and only if, they are satisfied for each of the p_{ij}. But, then,

$$p(+, +|A_1, B_1) = \sum_{i,j} q_{ij} p_{ij}(+, +|A_1, B_1) \tag{6}$$

is a convex sum of Hardy's probabilities in each two-qubit subspace [35]. As a convex sum, it is less or equal to the largest element in the combination, whose maximum value is known to be given by Eq. (1). This concludes the proof [36]. □

Hardy's test leads to self-testing.—It follows from the previous proof that $p(+, +|A_1, B_1)$ reaches its maximal value if and only if $p_{ij}(+, +|A_1, B_1)$ is maximal for every ij such that $q_{ij} \neq 0$. The following Lemma, proved in Refs. [10,11], states that only a very specific class of two-qubit states can lead to this maximal value:

Lemma 2.—Consider Hardy's test implemented in a two-qubit system, and let $A_0 = B_0 = |0\rangle\langle 0| - |1\rangle\langle 1|$. The probability p_{Hardy} reaches its maximal value if, and only if, the state of the system is

$$|\phi\rangle = a(|01\rangle + |10\rangle) + e^{i\theta}\sqrt{1 - 2a^2}|11\rangle, \tag{7}$$

and the other two measurements are $A_1 = B_1 = |+\rangle\langle+| - |-\rangle\langle-|$ with $|+\rangle = \frac{1}{\sqrt{1-a^2}}(\sqrt{1-2a^2}|0\rangle - e^{i\theta}a|1\rangle)$, $a = \sqrt{(3-\sqrt{5})/2}$, and θ is arbitrary.

In view of this, one can conjecture that, if the maximal value of p_{Hardy} is observed, the state must somehow be a direct sum of copies of $|\phi\rangle$. We proceed to prove that this is indeed the case:

Theorem 2.—If $p_{\text{Hardy}} = (5\sqrt{5} - 11)/2$ is observed in an ideal Hardy's test [i.e., together with Eq. (2)], then the state of the system is equivalent up to local isometries to $|\sigma\rangle_{AB} \otimes |\phi\rangle_{A'B'}$, where $|\phi\rangle$ is given in Eq. (7)and $|\sigma\rangle$ is an arbitrary bipartite state. In other words, the ideal Hardy's test constitutes a self-testing of $|\phi\rangle$.

Proof.—Without loss of generality, let us choose the eigenbases of A_0 and B_0 as the computational bases: $\Pi^i_{+|A_0} = |2i\rangle\langle 2i|$, $\Pi^i_{-|A_0} = |2i+1\rangle\langle 2i+1|$, $\Pi^j_{+|B_0} = |2j\rangle\langle 2j|$, and $\Pi^j_{-|B_0} = |2j+1\rangle\langle 2j+1|$. Then, by Lemma 2, $p_{ij}(+,+|A_1,B_1) = \text{Tr}(\rho_{ij}\Pi^i_{+|A_1} \otimes \Pi^j_{+|B_1}) = (5\sqrt{5}-11)/2$ if, and only if, $\rho_{ij} = |\phi_{ij}\rangle\langle\phi_{ij}|$, where

$$|\phi_{ij}\rangle = a(|2i, 2j+1\rangle + |2i+1, 2j\rangle) + e^{i\theta}\sqrt{1-2a^2}|2i+1, 2j+1\rangle, \quad (8)$$

and $a = \sqrt{(3-\sqrt{5})/2}$ and arbitrary θ. This way, a state $|\psi\rangle$ can lead to a maximal value of p_{Hardy} if, and only if, it is given by

$$|\psi\rangle = \bigoplus_{i,j} \sqrt{q_{ij}}\,|\phi_{ij}\rangle. \quad (9)$$

The coefficients q_{ij} are arbitrary probabilities that, by definition, are constrained to the form $q_{ij} = r_i s_j$, where r_i, $s_j \geq 0$ and $\sum_i r_i = \sum_j s_j = 1$. The angle θ cannot depend on the indices i, j because $\Pi^i_{+|A_1}$ is uniquely defined by θ (cf. Lemma 2), and, by definition, is independent of j; the same reasoning can be applied to $\Pi^j_{+|B_1}$. Now, following Ref. [5], we append local ancilla qubits prepared in the state $|00\rangle_{A'B'}$ and look for local isometries Φ_A and Φ_B such that

$$(\Phi_A \otimes \Phi_B)|\psi\rangle_{AB}|00\rangle_{A'B'} = |\sigma\rangle_{AB}|\phi\rangle_{A'B'}, \quad (10)$$

where $|\sigma\rangle$ is a bipartite *junk* state. This can, indeed, be achieved for $\Phi_A = \Phi_B = \Phi$ defined by the map

$$\Phi|2k, 0\rangle_{CC'} \mapsto |2k, 0\rangle_{CC'}, \quad (11a)$$

$$\Phi|2k+1, 0\rangle_{CC'} \mapsto |2k, 1\rangle_{CC'}, \quad (11b)$$

for both $C = A, B$.

Up to now, self-testing was known only for maximally entangled states (see, e.g., Ref. [5] and references therein). A parallel, independent work by Yang and Navascués [33] provides a very general approach to the self-testing of bipartite nonmaximally entangled states. Remarkably, though, our Hardy point is not detected by that test [37].

Hardy's experiment with realistic constraints.—Suppose now that the constraint probabilities [Eq. (2)] in Hardy's experiment are not exactly equal to zero. In this case, the local bound on Hardy's probability is no longer zero, either, in general, it is given by the following inequality [26,27]:

$$p(+,+|A_1,B_1) \leq p(+,+|A_0,B_0) + p(+,-|A_1,B_0) + p(-,+|A_0,B_1). \quad (12)$$

This inequality is a rewriting of the CH inequality [29], which is not surprising, since the CH inequality is the only relevant criterion for nonlocality in a scenario with two parties, two inputs, and two outcomes. In other words, as noticed in Ref. [28], Hardy's experiment turns out to be a study of the violation of the CH inequality under further constraints about the values of some probabilities.

Let us now set

$$p(+,+|A_0,B_0) \leq \epsilon, \quad (13a)$$

$$p(+,-|A_1,B_0) \leq \epsilon, \quad (13b)$$

$$p(-,+|A_0,B_1) \leq \epsilon, \quad (13c)$$

for some $\epsilon \geq 0$ [38]. The local bound on Hardy's probability becomes

$$p(+,+|A_1,B_1) \leq 3\epsilon. \quad (14)$$

For $\epsilon \geq \frac{1}{3}$, the bound is trivial and quantum physics certainly cannot violate it, while for $0 \leq \epsilon < \frac{1}{3}$, quantum physics may lead to a violation of the local bound. As before, we want to assess the maximal quantum violation in a device-independent scenario, i.e., without making any assumption on the Hilbert space dimension. The previously stated theorem cannot be extended, so we take a different approach: first, we use semidefinite programs to obtain an upper bound on Hardy's probability, using the method of Navascués, Pironio, and Acín [39]; second, by considering two-qubit systems, we obtain a value that is certainly achievable with quantum systems. By noticing that the values thus obtained coincide, we conclude that we have obtained the optimal value for Hardy's probability, and that this value can be reached with two-qubit systems.

In detail, let $\mathcal{Q}$ be the set of quantum joint probability distributions, that is, vectors of probabilities of the form [Eq. (4)], for all a, b, x, y. We compute a device-independent upper bound on Hardy's probability by optimizing it, not over quantum probabilities in the set $\mathcal{Q}$, but over a larger set of probabilities that is computationally tractable—as opposed to $\mathcal{Q}$, that still lacks a better characterization. This set is one of an infinite hierarchy of sets $\mathcal{Q}_1 \supset \mathcal{Q}_2 \supset \ldots \supset \mathcal{Q}_n \supset \ldots$, defined in terms of semidefinite programs [39,40], proven to converge to the quantum set, $\lim_{n\to\infty}\mathcal{Q}_n = \mathcal{Q}$. For several values of ϵ in the interval $0 \leq \epsilon \leq 1/3$, we optimize Hardy's probability over the set Q_3, enforcing the constraints [Eq. (13)]. The

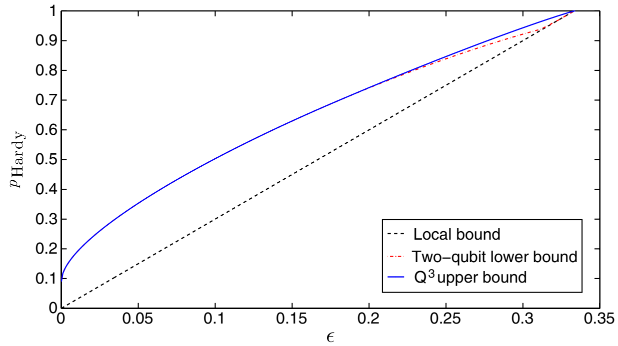

FIG. 2 (color online). Upper and lower bounds on maximum Hardy's probability, p_{Hardy}, in terms of the bound ϵ on the constraint probabilities. The solid (blue) line is the upper bound, computed from the set $\mathcal{Q}_3$; the dotted (red) line is the lower bound, computed from two-qubit systems; the dashed (black) line is the local bound.

implementation was done in MATLAB using semidefinite programming [41,42]. The results form the solid line in Fig. 2. For the lower bound, we consider the most general mixed states of two qubits and positive operator-value measure elements acting on those. The maximal value of the Hardy's probability is estimated using constrained non-linear optimization methods in MATLAB. These methods are not guaranteed to converge to global maxima, though, and are in fact rather sensitive to seed conditions; each point on the dotted line in Fig. 2 is the maximum obtained over 10^4 runs, with random initial seeds.

The computed lower and upper bounds for Hardy's probability differ, at most, by values of order 10^{-2}; in the region $\epsilon \lesssim 0.2$ (where any experiment that aims at implementing Hardy's test will have to be), this difference is of order 10^{-6}. This proves that there is no advantage in using higher-dimensional systems, as compared to two-qubit systems, even in the presence of imperfections.

Conclusion.—In this Letter, we prove that the maximum value of Hardy's probability found for two-qubit systems, $(5\sqrt{5} - 11)/2$, is the maximum one allowed by quantum theory, irrespective of the dimension of the system and of the measurements performed, that is, independent of the devices used. By showing that only a certain class of states can lead to such maximal value, we show that Hardy's test is, in fact, a self-testing protocol for such states. Extending the first results to a nonideal version of Hardy's test, where the constraint probabilities are no longer equal to zero, we compute device-independent upper bounds on Hardy's probability, in terms of the error parameter, and show that this bound is saturated by two-qubit systems.

Despite their fundamental importance, as the first proven analogue of Tsirelson's bound for Hardy's test, the results here presented also serve as a guideline for future experimental implementations, as they show that there is no advantage in using higher dimensional systems, as compared to two-qubit systems.

The authors thank Tobias Fritz, Sibasish Ghosh, and Marcelo Terra Cunha for discussions and comments. This work was supported by the National Research Foundation and the Ministry of Education, Singapore.

[1] J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).
[2] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Phys. Rev. Lett. **98**, 230501 (2007).
[3] E. Hänggi, R. Renner, and S. Wolf, in *Advances in Cryptology—EUROCRYPT 2010: 29th Annual International Conference on the Theory and Applications of Cryptographic Techniques, French Riviera, 2010*, edited by H. Gilbert (Springer, Berlin, Heidelberg, 2010), p. 216.
[4] C. E. Bardyn, T. C. H. Liew, S. Massar, M. McKague, and V. Scarani, Phys. Rev. A **80**, 062327 (2009).
[5] M. McKague, T. H. Yang, and V. Scarani, arXiv:1203.2976.
[6] R. Rabelo, M. Ho, D. Cavalcanti, N. Brunner, and V. Scarani, Phys. Rev. Lett. **107**, 050502 (2011).
[7] S. Pironio *et al.*, Nature (London) **464**, 1021 (2010).
[8] R. Colbeck and A. Kent, J. Phys. A **44**, 095305 (2011).
[9] L. Hardy, Phys. Rev. Lett. **68**, 2981 (1992).
[10] L. Hardy, Phys. Rev. Lett. **71**, 1665 (1993).
[11] S. Goldstein, Phys. Rev. Lett. **72**, 1951 (1994).
[12] C. Pagonis and R. Clifton, Phys. Lett. A **168**, 100 (1992).
[13] G. Kar, Phys. Rev. A **56**, 1023 (1997).
[14] J. Cereceda, Phys. Lett. A **327**, 433 (2004).
[15] D. Boschi, S. Branca, F. De Martini, and L. Hardy, Phys. Rev. Lett. **79**, 2755 (1997).
[16] Y.-C. Liang, R. W. Spekkens, and H. M. Wiseman, Phys. Rep. **506**, 1 (2011).
[17] R. Clifton and P. Neiman, Phys. Lett. A **166**, 177 (1992).
[18] S. Kunkri and S. K. Choudhary, Phys. Rev. A **72**, 022348 (2005).
[19] K. Seshdreesan and S. Ghosh, J. Phys. A **44**, 315305 (2011).
[20] M. Hillery, B. Yurke, and D. Stoler, Phys. Rev. A **63**, 062111 (2001).
[21] J. R. Torgerson, D Branning, C. H. Monken, and L. Mandel, Phys. Lett. A **204**, 323 (1995).
[22] G. Di Giuseppe, F. De Martini, and D. Boschi, Phys. Rev. A **56**, 176 (1997).
[23] W. T. M. Irvine, J. F. Hodelin, C. Simon, and D. Bouwmeester, Phys. Rev. Lett. **95**, 030401 (2005).
[24] A. Fedrizzi, M. P. Almeida, M. A. Broome, A. G. White, and M. Barbieri, Phys. Rev. Lett. **106**, 200402 (2011).
[25] G. Vallone, I. Gianani, E. B. Inostroza, C. Saavedra, G. Lima, A. Cabello, and P. Mataloni, Phys. Rev. A **83**, 042105 (2011).
[26] D. Mermin, Am. J. Phys. **62**, 880 (1994).
[27] A. Garuccio, Phys. Rev. A **52**, 2535 (1995).
[28] D. Braun and M.-S. Choi, Phys. Rev. A **78**, 032114 (2008).
[29] J. F. Clauser and M. Horne, Phys. Rev. D **10**, 526 (1974).
[30] B. S. Cirelson, Lett. Math. Phys. **4**, 93 (1980).
[31] D. Mayers and A. Yao, in *FOCS 98: Proceedings of the 39th Annual Symposium on Foundations of Computer Science* (IEEE Computer Society, Washington, DC, 1998), p. 503.

[32] D. Mayers and A. Yao, Quantum Inf. Comput. **4**, 273 (2004).
[33] T. H. Yang and M. Navascués (to be published).
[34] L. Masanes, Phys. Rev. Lett. **97**, 050503 (2006).
[35] It is important to note that for the maximum value of Eq. (6) to be reached, it is necessary that, for all i, j such that $q_{ij} \neq 0$, the dimension of both $\mathcal{H}^i$ and $\mathcal{H}^j$ be equal to two. This implies that the effective dimension d of the local Hilbert space $\mathcal{H}$ of the system is even, and that A_0, A_1, B_0, B_1 have exactly $d/2$ positive and negative eigenvalues.
[36] There is an alternative, simpler proof of the above theorem that consists, basically, in noticing that any probability distribution that maximizes Hardy's probability is an extremal point of the set of quantum probability distributions. According to Ref. [34], every extremal point, in this scenario, can be obtained from projective measurements on two-qubit systems, thus proving the stated result. The reason we opted for presenting the extensive proof is that it leads to interesting insights about the states that lead to such maximal violation, as discussed below. This proof cannot be extended to the nonideal scenario we later consider due to the fact that, in that scenario, it is not clear wether the points reaching maximal Hardy's probability are extremal or not.
[37] At least not up to the $\mathcal{Q}_2$ step of the hierarchy [T. H. Yang (private communication)].
[38] Notice that, if no-signaling holds, then $p(+,+|A_0,B_0) = \epsilon$ implies $p(+|A_0) \geq \epsilon$ and $p(-,+|A_0,B_1) = \epsilon$ implies $p(-|A_0) \geq \epsilon$. Therefore, $\epsilon \leq \frac{1}{2}$. But the region of interest is in fact $\epsilon \leq \frac{1}{3}$ as explained just below.
[39] M. Navascues, S. Pironio, and A. Acín, New J. Phys. **10**, 073013 (2008).
[40] A. Doherty, Y.-C. Liang, B. Toner, and S. Wehner, in *Proceedings of the 23rd Annual IEEE Conference on Computational Complexity* (IEEE Computer Society, Washington, DC, 2008), p. 199.
[41] J. F. Sturm, Opt. Meth. Softw. **11**, 625 (1999).
[42] J. Löfberg, in *YALMIP: A Toolbox for Modeling and Optimization in MATLAB, Proceedings of the CACSD Conference, Taipei, Taiwan, 2004* (IEEE, Piscataway, NJ, 2004).