

Chiral Kinetic Theory

M. A. Stephanov and Y. Yin

Department of Physics, University of Illinois, Chicago, Illinois 60607, USA

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We derive the nonequilibrium kinetic equation describing the motion of chiral massless particles in the regime where it can be considered classically. We show that the Berry monopole which appears at the origin of the momentum space due to level crossing is responsible for the chiral magnetic and vortical effects.

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Introduction.—The generation of nondissipative currents in a chiral (parity violating) system in response to an external magnetic field has attracted significant interest recently. Such an effect, noted earlier in different contexts in Refs. [1,2], has been recently proposed as an intriguing explanation for the charge-dependent correlations in heavy-ion collisions in Refs. [3,4] and termed the chiral magnetic effect (CME). It has been also shown recently in Ref. [5] that the hydrodynamics of chiral systems with an anomaly requires the presence of such currents, as well as currents induced by the vorticity of flow—the chiral vortical effect (CVE), discovered earlier in a microscopic calculation in an astrophysical context in Ref. [6], and rediscovered recently in gauge-gravity duality calculations in Refs. [7,8].

The interesting applications of these chiral transport effects involve highly nonequilibrium conditions, such as those arising in the early stages of the heavy-ion collisions, when the magnetic fields created by passing ions are still strong. However, derivations of these effects have been mostly done assuming thermal and chemical equilibrium. The aim of this Letter is to address this shortcoming of theory.

A natural framework to study nonequilibrium conditions is a kinetic theory. As any useful theory, it has limitations, such as assumption of the classical motion between collisions and the weakness of the coupling. Nevertheless, a kinetic description would undoubtedly be an important step for our understanding of the chiral transport phenomena.

Most of the ingredients of the approach presented here can be found in the literature on the physics of geometric phases introduced by Berry in Ref. [9]. The relevant classical equations of motion were introduced in Ref. [10] (see also Ref. [11] for a review). The kinetic equation in the presence of the Berry curvature has been studied, e.g., in Refs. [12,13]. The most recent and closely relevant applications include Refs. [14,15]. Unrelated to the above so far, a very important step towards a kinetic description of the CME and CVE was made recently in Ref. [16].

Putting these ingredients together we derive the desired nonequilibrium expressions for the CME and CVE which

are, to the extent of our knowledge, new. The key observation of the present Letter is that for Weyl fermions the Berry curvature, being the field of a monopole, leads directly to the CME. We also point out that the CVE can be similarly understood by simply replacing the Lorentz force due to the magnetic field by the Coriolis force. In the following, we shall present a reasonably self-contained derivation of these results using a formalism somewhat complementary to traditional approaches. This will allow us also to make connections to other field-theoretical concepts (such as Abelian projections) more familiar in the particle theory context.

Kinetic equation.—The kinetic equation describes the motion of particles in the regime where collisions are rare enough that motion between collisions is classical. In terms of the distribution function $f(t, \mathbf{x}, \mathbf{p})$ the equation reads

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \dot{\mathbf{x}} + \frac{\partial f}{\partial \mathbf{p}} \dot{\mathbf{p}} = C[f]. \quad (1)$$

We think of a “cloud” of particles each of which follows its classical trajectory $\mathbf{x}(t)$, $\mathbf{p}(t)$. As a result, the distribution evolves with time in such a way that if one follows a local volume occupied by a set of particles along the trajectory, the number of particles in it can only be changed by collisions.

The CME is known to be closely related to the chiral anomaly [2,3,5]. On the other hand, it is clear from the description above that the number of particles in the phase space *cannot change*. How could a kinetic equation account for anomalous particle number nonconservation? In other words, how can a *classical* equation account for a *quantum* anomaly? As we shall see below, the answer is, in two words: Berry monopole.

Path integral and $U(2)$ gauge invariance.—Consider the Hamiltonian for a Weyl particle:

$$H = \boldsymbol{\sigma} \cdot \mathbf{p}. \quad (2)$$

For each momentum \mathbf{p} it represents a two-state system with energy gap $2|\mathbf{p}|$.

It is more straightforward to obtain the classical limit in the path integral formulation rather than in the canonical formulation of quantum mechanics usually employed to

describe the Berry connection. Consider the transition amplitude between the two spin states i and f . By inserting the sums over complete sets of eigenstates of coordinates and spin, $|\mathbf{x}, s\rangle$, and momenta and spin, $|\mathbf{p}, \lambda\rangle$, the amplitude can be written as a path integral

$$\langle f | e^{iH(t_f - t_i)} | i \rangle = \left[\int \mathcal{D}\mathbf{x} \mathcal{D}\mathbf{p} \mathcal{P} \exp \left\{ i \int_{t_i}^{t_f} (\mathbf{p} \cdot \dot{\mathbf{x}} - \boldsymbol{\sigma} \cdot \mathbf{p}) dt \right\} \right]_{fi}, \quad (3)$$

where we need to take a matrix element $[\dots]_{fi}$ of the path-ordered product of the matrices $\exp\{-i\boldsymbol{\sigma} \cdot \mathbf{p} \Delta t\}$ over each path $\mathbf{x}(t)$, $\mathbf{p}(t)$ in the phase space. These matrices can be thought of as describing the rotation of the state of the particle in the spin space as it moves along.

The massless particles we describe have only one helicity state (the opposite helicity state corresponds to an antiparticle). In order to consider classical motion of such a particle we need to diagonalize the matrix in the helicity basis and then apply the usual method of stationary phase to determine the classical trajectory. This diagonalization can be done at each point on the trajectory using unitary matrix $V_{\mathbf{p}}$ such that

$$V_{\mathbf{p}}^\dagger \boldsymbol{\sigma} \cdot \mathbf{p} V_{\mathbf{p}} = |\mathbf{p}| \sigma_3. \quad (4)$$

If the values of momenta at two neighboring points t_1 and t_2 are \mathbf{p}_1 and \mathbf{p}_2 , we insert identity matrices between the exponential factors in the following way:

$$\begin{aligned} & \dots V_{\mathbf{p}_2} V_{\mathbf{p}_2}^\dagger \exp\{-i\boldsymbol{\sigma} \cdot \mathbf{p}_2 \Delta t\} V_{\mathbf{p}_2} V_{\mathbf{p}_2}^\dagger V_{\mathbf{p}_1} V_{\mathbf{p}_1}^\dagger \\ & \quad \times \exp\{-i\boldsymbol{\sigma} \cdot \mathbf{p}_1 \Delta t\} V_{\mathbf{p}_1} V_{\mathbf{p}_1}^\dagger \dots \\ & = \dots V_{\mathbf{p}_2} \exp\{-i|\mathbf{p}_2| \sigma_3 \Delta t\} V_{\mathbf{p}_2}^\dagger V_{\mathbf{p}_1} \\ & \quad \times \exp\{-i|\mathbf{p}_1| \sigma_3 \Delta t\} V_{\mathbf{p}_1}^\dagger \dots \end{aligned} \quad (5)$$

If the $\Delta \mathbf{p} \equiv \mathbf{p}_2 - \mathbf{p}_1$ is small, we can write the extra unitary rotation between the two neighboring exponents as

$$V_{\mathbf{p}_2}^\dagger V_{\mathbf{p}_1} \approx \exp(-i\hat{\mathbf{a}}_{\mathbf{p}} \cdot \Delta \mathbf{p}), \quad \text{where } \hat{\mathbf{a}}_{\mathbf{p}} = iV_{\mathbf{p}}^\dagger \nabla_{\mathbf{p}} V_{\mathbf{p}} \quad (6)$$

is a Hermitian 2×2 matrix.

Performing the above diagonalization along the whole trajectory and assembling the exponents into the path integral one obtains an alternative expression for the amplitude in Eq. (3):

$$\langle f | e^{iH(t_f - t_i)} | i \rangle = \left[V_{\mathbf{p}_f} \int \mathcal{D}\mathbf{x} \mathcal{D}\mathbf{p} \mathcal{P} \exp \left\{ i \int_{t_i}^{t_f} (\mathbf{p} \cdot \dot{\mathbf{x}} - |\mathbf{p}| \sigma_3 - \hat{\mathbf{a}}_{\mathbf{p}} \cdot \dot{\mathbf{p}}) dt \right\} V_{\mathbf{p}_i}^\dagger \right]_{fi}. \quad (7)$$

If we did not insist on diagonalizing the matrix $\boldsymbol{\sigma} \cdot \mathbf{p}$, we could have chosen an arbitrary $U(2)$ rotation, say $V_{\mathbf{p}} U_{\mathbf{p}}$, instead of $V_{\mathbf{p}}$. This results in a local ‘‘gauge transformation’’ of the ‘‘action’’ such that

$$\begin{aligned} -|\mathbf{p}| \sigma_3 & \rightarrow -|\mathbf{p}| U_{\mathbf{p}}^\dagger \sigma_3 U_{\mathbf{p}}, \\ \hat{\mathbf{a}}_{\mathbf{p}} & \rightarrow U_{\mathbf{p}}^\dagger \hat{\mathbf{a}}_{\mathbf{p}} U_{\mathbf{p}} + iU_{\mathbf{p}}^\dagger \nabla_{\mathbf{p}} U_{\mathbf{p}}. \end{aligned} \quad (8)$$

This gauge freedom corresponds to the free choice of the phase and spin quantization direction for the momentum states: $|\mathbf{p}, s\rangle \rightarrow U_{\mathbf{p}} |\mathbf{p}, s\rangle$ along the trajectory. Clearly this choice only affects the expression for the amplitude in Eq. (7), and not the value of the amplitude itself. We use this redundancy of description to choose the helicity basis at each \mathbf{p} . This choice diagonalizes $\boldsymbol{\sigma} \cdot \mathbf{p}$ and enables us to take the classical limit.

Abelian projection and Berry monopole.—Fixing this non-Abelian $U(2)$ gauge freedom by diagonalizing the Hamiltonian is mathematically similar to the Abelian projection introduced by ’t Hooft in Ref. [17]. In the classical regime the contribution of the transitions caused by the off-diagonal components of $\hat{\mathbf{a}}_{\mathbf{p}}$ is negligible (in Ref. [17] the ‘‘non-Abelian’’ part of the gauge field is nonpropagating due to confinement). Suppressing these off-diagonal components, we still have a $U(1) \times U(1)$ gauge freedom corresponding to selecting arbitrarily the complex phases for the helicity eigenstates at each momentum. Focusing on helicity +1 we can denote the corresponding diagonal component $[\hat{\mathbf{a}}_{\mathbf{p}}]_{11} \equiv \mathbf{a}_{\mathbf{p}}$. Then the classical action for the helicity +1 particle becomes

$$I = \int_{t_i}^{t_f} (\mathbf{p} \cdot \dot{\mathbf{x}} - |\mathbf{p}| - \mathbf{a}_{\mathbf{p}} \cdot \dot{\mathbf{p}}) dt. \quad (9)$$

The classical, or adiabatic, approximation will break down when the two eigenvalues of the Hamiltonian are degenerate, i.e., at $\mathbf{p} = 0$. As we shall see, this point is the source of the effects of the quantum anomaly.

As in ’t Hooft’s original application of the Abelian projection, even if the non-Abelian field $\hat{\mathbf{a}}_{\mathbf{p}}$ is a pure gauge, Eq. (6), the Abelian component $[\hat{\mathbf{a}}_{\mathbf{p}}]_{11} \equiv \mathbf{a}_{\mathbf{p}}$ is nontrivial. Finding the unitary matrix $V_{\mathbf{p}}$ in Eq. (4) and calculating $\hat{\mathbf{a}}_{\mathbf{p}}$ in Eq. (6), one obtains the well-known result that the corresponding Abelian field $\mathbf{a}_{\mathbf{p}}$ is the field of a ‘‘monopole’’ [18,19] at $|\mathbf{p}| = 0$. Of course, the physical amplitude cannot depend on the gauge choice in Eq. (9). We expect physical observables to depend only on the Abelian field strength $\mathbf{b} = \nabla_{\mathbf{p}} \times \mathbf{a}_{\mathbf{p}}$. One finds

$$\mathbf{b} = \frac{\hat{\mathbf{p}}}{2|\mathbf{p}|^2}, \quad \text{where } \hat{\mathbf{p}} \equiv \frac{\mathbf{p}}{|\mathbf{p}|}. \quad (10)$$

Equations of motion.—Before we write the classical equations of motion, let us quantify the conditions of their applicability. The classical, or adiabatic, approximation requires the off-diagonal components of $\hat{\mathbf{a}}_{\mathbf{p}} \cdot \dot{\mathbf{p}}$ in Eq. (7) to be small compared to the energy gap $2|\mathbf{p}|$. This means the forces, $\dot{\mathbf{p}}$, on the particle cannot be too strong. For example, if the particle moves in a magnetic field B : $B \ll |\mathbf{p}|^2$, where we used $|\hat{\mathbf{a}}_{\mathbf{p}}| \sim 1/|\mathbf{p}|$ [cf. Eq. (10)]. This condition is obvious physically, since particles with

momenta as low as the momenta on the lowest Landau orbit cannot be treated classically.

It is easy to couple the classical particle described by the action in Eq. (9) to an external electromagnetic field given by scalar and vector potentials Φ and \mathbf{A} . By variations of the resulting action

$$I = \int_{t_i}^{t_f} (\mathbf{p} \cdot \dot{\mathbf{x}} + \mathbf{A} \cdot \dot{\mathbf{x}} - \Phi - |\mathbf{p}| - \mathbf{a}_p \cdot \dot{\mathbf{p}}) dt, \quad (11)$$

one obtains the desired equations of motion (cf. Refs. [10,11]):

$$\dot{\mathbf{x}} = \hat{\mathbf{p}} + \dot{\mathbf{p}} \times \mathbf{b}; \quad (12)$$

$$\dot{\mathbf{p}} = \mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}. \quad (13)$$

Without the Berry flux \mathbf{b} , these equations are the familiar equations for the velocity of a massless particle and the Lorentz force. Without electromagnetic field the Berry curvature \mathbf{b} drops out of the equations, because $\dot{\mathbf{p}} = 0$.

Substituting Eq. (13) into Eq. (12) and solving for $\dot{\mathbf{x}}$ one finds

$$\sqrt{G}\dot{\mathbf{x}} = \hat{\mathbf{p}} + \mathbf{E} \times \mathbf{b} + \mathbf{B}(\hat{\mathbf{p}} \cdot \mathbf{b}); \quad (14)$$

$$\sqrt{G}\dot{\mathbf{p}} = \mathbf{E} + \hat{\mathbf{p}} \times \mathbf{B} + \mathbf{b}(\mathbf{E} \cdot \mathbf{B}). \quad (15)$$

Here, $G = (1 + \mathbf{b} \cdot \mathbf{B})^2$ is the determinant of the 6×6 matrix of coefficients in Eqs. (12) and (13) for $\dot{\mathbf{x}}$ and $\dot{\mathbf{p}}$. Substituting into Eq. (1) we can then obtain the desired kinetic equation for the distribution function of such particles in the phase space.

Chiral magnetic effect.—It is important to take account of the fact that the invariant measure of the phase-space integration is given by $\sqrt{G}d^3\mathbf{x}d^3\mathbf{p}/(2\pi)^3$; see, e.g., Ref. [20]. In particular, one can check using equations of motion (14) and (15), and Maxwell equations $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{E} = \partial\mathbf{B}/\partial t$ that this measure obeys the Liouville equation

$$\frac{\partial}{\partial t}\sqrt{G} + \frac{\partial}{\partial \mathbf{x}}(\sqrt{G}\dot{\mathbf{x}}) + \frac{\partial}{\partial \mathbf{p}}(\sqrt{G}\dot{\mathbf{p}}) = 2\pi\mathbf{E} \cdot \mathbf{B}\delta^3(\mathbf{p}), \quad (16)$$

where the last term is due to the Berry monopole $\nabla_{\mathbf{p}} \cdot \mathbf{b} = 2\pi\delta^3(\mathbf{p})$, Eq. (10). The last term is the effect of the quantum anomaly which “injects” particle number violation into our otherwise classical description. It is notable that this term is localized at $\mathbf{p} = 0$, where the classical description must break down due to level crossing.

The current density is given by $\mathbf{j} = \int_{\mathbf{p}} \sqrt{G}f\dot{\mathbf{x}}$, where $\int_{\mathbf{p}} \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3}$. Using Eq. (14) we find

$$\mathbf{j} = \int_{\mathbf{p}} \sqrt{G}f\dot{\mathbf{x}} = \int_{\mathbf{p}} f\hat{\mathbf{p}} + \mathbf{E} \times \int_{\mathbf{p}} f\mathbf{b} + \mathbf{B} \int_{\mathbf{p}} f(\hat{\mathbf{p}} \cdot \mathbf{b}). \quad (17)$$

The first term gives the usual current, while the second is the anomalous Hall current. Both vanish in a state with isotropic momentum distribution, such as the equilibrium

state. The last term is the desired nonequilibrium expression of the CME.

Using notations $E = |\mathbf{p}|$ and an overbar to denote the average over the unit sphere of directions of vector $\hat{\mathbf{p}}$ we can write

$$\mathbf{j}_{\text{CME}} = \mathbf{B} \int_{\mathbf{p}} f(\hat{\mathbf{p}} \cdot \mathbf{b}) = \frac{\mathbf{B}}{4\pi^2} \int_0^\infty \overline{f(E, \hat{\mathbf{p}})} dE. \quad (18)$$

This equation agrees with the result conjectured in Ref. [16] for an isotropic distribution. In the case of the Fermi-Dirac distribution it reproduces the well-known results (such as $\mathbf{j}_{\text{CME}} = \mu\mathbf{B}/(2\pi)^2$ at zero temperature).

Chiral anomaly.—To find the effect of the electromagnetic anomaly we calculate the 4-divergence of the particle number current in Eq. (17). It is illuminating to begin the discussion by introducing the $6 + 1$ phase space current $(\rho, \rho\dot{\mathbf{x}}, \rho\dot{\mathbf{p}})$, where $\rho = \sqrt{G}f$ obeys the continuity equation with a source [21]

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\dot{\mathbf{x}})}{\partial \mathbf{x}} + \frac{\partial(\rho\dot{\mathbf{p}})}{\partial \mathbf{p}} = 2\pi\mathbf{E} \cdot \mathbf{B}f\delta^3(\mathbf{p}), \quad (19)$$

which follows from Eq. (1) and Eq. (16). Integrating over momentum \mathbf{p} we obtain

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{j} = \frac{1}{4\pi^2} \mathbf{E} \cdot \mathbf{B}f_0, \quad (20)$$

where, as in Eq. (17), $(n, \mathbf{j}) = \int_{\mathbf{p}} (\rho, \rho\dot{\mathbf{x}})$ is the $3 + 1$ space-time current density and f_0 is the value of the distribution function f at $\mathbf{p} = 0$. For the Fermi-Dirac distribution at zero temperature and nonzero chemical potential, $f_0 = 1$ and we reproduce the standard expression of the electromagnetic anomaly.

Strictly speaking the above calculation is not completely legitimate because we integrated over the whole momentum space, including the singular point $\mathbf{p} = 0$, where the classical description is not applicable. The way to think about this equation is to exclude the region $|\mathbf{p}| < \Delta$ around the singularity. The value of Δ must be large enough so that the classical description applies outside of it ($\Delta \gg \sqrt{B}$).

Then, in the classical region $|\mathbf{p}| > \Delta$, the $6 + 1$ phase space current $(\rho, \rho\dot{\mathbf{x}}, \rho\dot{\mathbf{p}})$ obeys the continuity equation. In other words, the particles, in the absence of collisions, cannot be created or destroyed in the classical region. They can only enter or exit the region through the boundary of the region at $|\mathbf{p}| = \Delta$. Integrating the continuity equation over the classical region $|\mathbf{p}| > \Delta$ and defining the $3 + 1$ current density in the classical region only $(n_\Delta, \mathbf{j}_\Delta) = \int_{|\mathbf{p}| > \Delta} (\rho, \rho\dot{\mathbf{x}})$ we find that the nonconservation of the $3 + 1$ space-time current is matched by the momentum-space flux into the classical region through the boundary at $|\mathbf{p}| = \Delta$:

$$\frac{\partial n_\Delta}{\partial t} + \nabla \cdot \mathbf{j}_\Delta = \int \frac{dS_\Delta}{(2\pi)^3} \cdot \mathbf{J}_p, \quad (21)$$

where the flux density is given by

$$\mathbf{J}_p \equiv \rho \dot{\mathbf{p}} = (\mathbf{E} + \hat{\mathbf{p}} \times \mathbf{B})f + 2\pi\mathbf{E} \cdot \mathbf{B}f \frac{\hat{\mathbf{p}}}{4\pi|\mathbf{p}|^2}. \quad (22)$$

The first term on the right-hand side is due to acceleration of the particles on the boundary $|\mathbf{p}| = \Delta$ which moves them in or out of the classical region. This term gives a negligible contribution to the total flux in Eq. (21) if Δ is small enough that the variation of f over the boundary can be neglected. The total flux from the last term, however, tends to a finite limit when $\Delta \rightarrow 0$, which is given by Eq. (20). The origin of this net flux is the anomaly which operates, as is well known, at the point of level crossing $\mathbf{p} = 0$, lying inside the region $|\mathbf{p}| < \Delta$, where the motion of particles must be treated fully quantum mechanically.

Chiral vortical effect.—To describe the CVE we need to realize that, unlike the external magnetic field \mathbf{B} , which we can set directly, the vorticity $\boldsymbol{\omega}$ is a property of the flow of particles, which is indirectly controlled by external fields and initial conditions. Moreover, the definition of vorticity involves the hydrodynamic limit, which puts additional conditions on flow. However, we can generalize the vorticity to nonequilibrium flows in the following way. We can decide to observe a given local fluid element in a co-moving frame, which will have to rotate with angular velocity $\boldsymbol{\omega}$ with respect to the laboratory. The particles will experience additional noninertial forces in this frame. At this point we can generalize the problem to nonequilibrium by asking what additional currents such noninertial forces induce.

To linear order the only such force is the Coriolis force:

$$\dot{\mathbf{p}} = 2|\mathbf{p}|\boldsymbol{\omega} \times \hat{\mathbf{x}} + \mathcal{O}(\omega^2). \quad (23)$$

(This classical result can be also verified by considering the Weyl Hamiltonian in the rotating frame.) The effect of this force is the same as of a “magnetic field” $\mathbf{B} \rightarrow 2|\mathbf{p}|\boldsymbol{\omega}$. Making a corresponding substitution in Eq. (15) we arrive at the following equation for the nonequilibrium generalization of the CVE:

$$\mathbf{j}_{\text{CVE}} = \boldsymbol{\omega} \int_p 2|\mathbf{p}|f(\hat{\mathbf{p}} \cdot \mathbf{b}) = \frac{\boldsymbol{\omega}}{4\pi^2} \int_0^\infty \overline{f(E, \hat{\mathbf{p}})} 2E dE. \quad (24)$$

This result is also in agreement with Ref. [16] for isotropic distribution, and reduces to $\mathbf{j}_{\text{CVE}} = \mu^2 \boldsymbol{\omega} / (2\pi)^2$ for the well-known case of the Fermi-Dirac distribution.

Conclusion.—We presented a kinetic description of the chiral magnetic and chiral vortical effects given by the kinetic equation (1) with equations of motion (12) and (13). Although these equations are ubiquitous in the condensed matter literature on the effects of the Berry curvature, to our knowledge, their relationship to the chiral magnetic and chiral vortical effects has not been appreciated until now. The key observation that the Berry curvature for the Weyl Hamiltonian is sourced by a monopole at

$|\mathbf{p}| = 0$ leads directly to the general nonequilibrium expressions for the chiral magnetic and chiral vortical effects in Eqs. (18) and (24) which reproduce all known results in equilibrium.

The presence of the monopole singularity in the momentum space also provides a natural mechanism by which anomaly can change the particle number in an otherwise classical system. The classical description breaks down in the region surrounding the singularity at $\mathbf{p} = 0$ of the size of order of the typical momentum in the lowest Landau orbit. The net particle creation occurs by the purely quantum effect of anomaly (level crossing) inside this nonclassical region. The net flux of the particles into the classical region is then given by Eq. (21), which can serve as a boundary condition for the kinetic equation in the classical region.

It would be interesting to use the results obtained here to investigate the consequences of nonequilibrium for the chiral transport effects in heavy-ion collisions [3,4,22]. We leave this and other applications to further study.

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