Exact Results for Classical Casimir Interactions: Dirichlet and Drude Model in the Sphere-Sphere and Sphere-Plane Geometry

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Analytic expressions that describe Casimir interactions over the entire range of separations have been limited to planar surfaces. Here we derive analytic expressions for the classical or high-temperature limit of Casimir interactions between two spheres (interior and exterior configurations), including the sphere-plane geometry as a special case, using bispherical coordinates. We consider both Dirichlet boundary conditions and metallic boundary conditions described by the Drude model. At short distances, closed-form expansions are derived from the exact result, displaying an intricate structure of deviations from the commonly employed proximity force approximation.

The collective action of fluctuation "van der Waals" forces between individual atoms leads to forces between macroscopic surfaces [1,2]. The fluctuations can be of thermal [3] and/or quantum [4] nature. An important characteristic of these forces is their nonadditivity, which complicates the study of Casimir interactions in the thermodynamic limit involving a macroscopic number of particles. An important exception is the interaction of *planar, parallel* surfaces where symmetry allows for an exact solution, the so-called Lifshitz formula [5], which is a landmark in the physics of fluctuation forces. More complicated shapes have been studied at sufficiently short surface separations by the proximity force approximation (PFA) that is based on the Lifshitz formula, treating curvature by summing over planar surface elements [6]. The thermal Casimir interaction between spherical particles in a critical fluid has been computed by using conformal invariance for nearly touching and widely separated particles [7]. More recently, a number of analytical and numerical approaches have been developed to deal with the nonadditivity, most notably implementations of concepts from scattering theory [8–10]. However, it remains unclear to what extent and precision these approaches can handle the practically important limit of short distances. Attempts to obtain analytic predictions beyond the PFA have been limited to computations in first-order corrections from a gradient expansion [11–13].

The difficulty to compute deviations from the simple PFA can be traced back to the long-range nature of fluctuation forces which is poorly treated by the PFA. With increasing spatial dimensions, fluctuations decay more strongly, and the PFA is expected to become more reliable. The situation resembles that in statistical mechanics when phase transitions are studied by mean field theory, treating fluctuations poorly. In this context, the exact solution of the two-dimensional Ising model [14] helped in the understanding of phase transitions and inspired several developments in the theory of critical phenomena and related fields. This demonstrates the importance of exact solutions for a better understanding of complex phenomena. In particular, several forms for nonadditivity-induced deviations from the PFA have been hypothesized before, depending on boundary conditions and temperature [9,15,16]. Here we derive an exact solution for the interaction between spheres of different radii that displays a much richer structure for the Casimir interaction in the high-temperature limit than previously assumed.

In this Letter, we obtain analytic expressions for the high-temperature limit of the Casimir interactions between two spheres (interior and exterior configurations), including the sphere-plane geometry as a special case. We consider both a scalar field with Dirichlet boundary conditions and metallic boundary conditions described by the Drude model. The key tool of our approach is a bispherical coordinate system [17], which allows for a simple solution of the scattering problem of two spheres in the static limit (Laplace equation). The high-temperature sphere-plate problem has been analyzed before in the large-distance limit [18] and by large-scale numerics at short distances, including up to 5000 partial wave orders [15]. Our exact solution is universal, i.e., material independent, and displays full agreement with these two limiting cases. We derive from the exact result a closed-form short-distance expansion that reveals an intricate structure of deviations from the PFA. It turns out that, for Dirichlet boundary conditions, the classical Casimir *force* can be expanded as a Laurent series for small surface-to-surface separations L. For Drude metallic boundary conditions, the structure of the corrections to PFA is more complicated than suspected before [15], as it involves a double logarithm of L, as well

as powers of $\ln L$ multiplied by powers of L. In both cases the leading correction to the PFA energy is represented by a term proportional to $\ln L$, with a common coefficient that had been computed earlier [11] by using a recently proposed gradient expansion of the Casimir energy [12,13]. However, for experimentally accessible separations, the interaction of Drude metals is dominated entirely by the double logarithmic term.

Geometry, coordinates, and eigenfunctions.—We consider two spheres of radii R_1 and R_2 with surface-to-surface distance L. This geometry is conveniently parametrized in bispherical coordinates (μ, η, φ) [17], defined by $(x, y, z) = a(\sin\eta\cos\varphi, \sin\eta\sin\varphi, \sinh\mu)/(\cosh\mu - \cos\eta)$, where $z = \pm a$ are the foci of the two spheres defined by $\mu = \mu_1 > 0$ and $\mu = \mu_2 < 0$, respectively; see Fig. 1. The spheres have radii $R_1 = a/\sinh\mu_1$ and $R_2 = -a/\sinh\mu_2$ and distances $L_1 = a\coth\mu_1$ and $L_2 = -a\coth\mu_2$ from the origin. The center-to-center distance is $d = L_1 + L_2$. It is useful to express the μ_α in terms of the natural geometrical scales: $\mu_1 = \text{arccosh}\lambda_1$, $\mu_2 = -\text{arccosh}\lambda_2$ with $\lambda_\alpha = [L^2 + 2(L + R_\alpha)(R_1 + R_2)]/[2R_\alpha(L + R_1 + R_2)]$. We are interested in the classical Casimir interaction which is determined by the scattering of the spheres at zero frequency and is given by the Laplace equation. The latter is separable in bispherical coordinates, and its Green's function can be expanded as $G_0(\mu, \eta, \varphi; \mu', \eta', \varphi') = \sum_{l=0}^\infty \sum_{m=-l}^l \phi_{lm}^{\text{reg}}(\mu_<, \eta, \varphi)\phi_{lm}^{\text{out}*}(\mu_>, \eta', \varphi')$, where $\mu_{<(>)}$ is the smaller (larger) of μ and μ' and the regular and outgoing eigenfunctions are
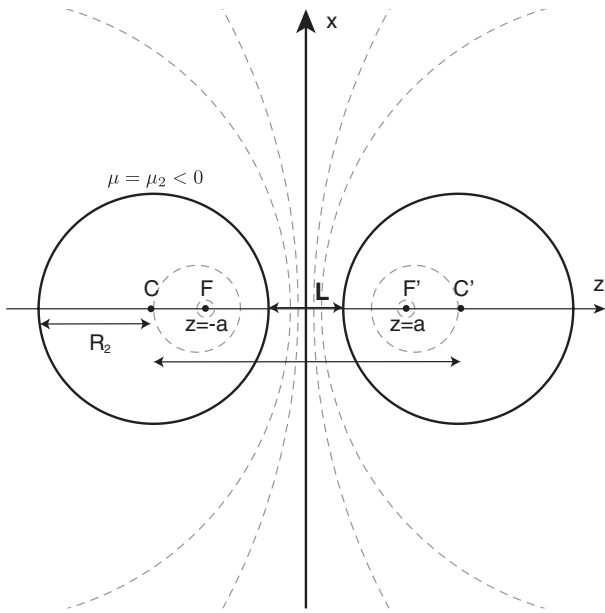

FIG. 1. Geometry of two spheres and a sphere plate. Shown are the centers (C, C') of the spheres and their foci (F, F'). The dashed curves correspond to curves of constant bispherical coordinate μ with $\mu = \mp\pi, \mp\pi/2, \mp\pi/10, \mp\pi/20$ with the upper (lower) sign for $z < 0$ $(z > 0)$, beginning with the smallest sphere.

$$\phi_{lm}^{\text{reg/out}} = \sqrt{\frac{\cosh\mu - \cos\eta}{a(2l+1)}} Y_{lm}(\eta, \varphi) e^{\pm(l+1/2)\mu} \quad (1)$$

for $l \geq 0$, $m = -l, \ldots, l$. Scattering solutions for various boundary conditions can be expanded in these eigenfunctions, leading to the zero-frequency (static) scattering amplitude $T_{lml'm'}^{(\alpha)}$ of sphere α by which we denote the matrix elements of the operator $\hat{T}^{(\alpha)}$ in the bispherical basis. The Casimir energy is expressed in the scattering approach [8] in terms of the scattering amplitudes and translation operators that translate the scattering solution from the coordinate of one object to the one of the other object. Here, for both spheres the scattering amplitude is expressed in the same bispherical coordinate system and hence the translation operators become the identity operators, yielding for the classical Casimir energy (or zeroth-order Matsubara term)

$$E = \frac{k_B T}{2} \ln \det[1 - \hat{N}] \quad \text{with} \quad \hat{N} = \hat{T}^{(1)}\hat{T}^{(2)}. \quad (2)$$

It is important to notice that the energy depends only on the equivalence class $[[N]]$ formed by all matrices that represent the operator $\hat{N}$; i.e., two elements N and MNM^{-1} of the class differ by an (invertible) similarity transformation M. In the following, we denote by $[[\ldots]]$ the equivalence class of the matrix enclosed by the brackets.

Dirichlet boundary conditions.—The scattering amplitudes for Dirichlet boundary conditions $\phi(\mu = \mu_\alpha) = 0$ on the spheres follows from Eq. (1) [8]. They assume the simple diagonal and m-independent form $T_{lml'm'}^{(\alpha)} = -\exp[\mp(2l+1)\mu_\alpha]\delta_{ll'}\delta_{mm'}$ with $-$ for $\alpha = 1$ and $+$ for $\alpha = 2$. Hence, the determinant in Eq. (2) can be easily computed, and we get for the classical Casimir energy of two Dirichlet spheres the exact result

$$E^{(\text{D})} = \frac{k_B T}{2} \sum_{l=0}^\infty (2l+1) \ln[1 - Z^{2l+1}], \quad (3)$$

which depends only via the single variable $Z = [\lambda_1 + \sqrt{\lambda_1^2 - 1}]^{-1}[\lambda_2 + \sqrt{\lambda_2^2 - 1}]^{-1}$ on the geometrical scales. For $|Z| < 1$, including the range of physically meaningful parameters, the energy $E^{(\text{D})}$ is an analytic function [19]. Large distances L correspond to small Z with $Z = R_1 R_2/L^2 + \mathcal{O}(L^{-3})$ and small distances to Z close to unity with $Z = 1 - \sqrt{2L/R_1 + 2L/R_2} + \mathcal{O}(L)$.

We note that the configuration of two spheres with one (R_1) inside the other (R_2) [20,21] can be computed exactly by the same method. The classical energy is given again by Eq. (3) but now with $Z = [\lambda_2 + \sqrt{\lambda_2^2 - 1}]/[\lambda_1 + \sqrt{\lambda_1^2 - 1}]$ with $\lambda_1 = [-L^2 + 2(L + R_1)(-R_1 + R_2)]/[2R_1(-L - R_1 + R_2)] > \lambda_2 = [L^2 + 2(-L + R_2)(-R_1 + R_2)]/[2R_2(-L - R_1 + R_2)]$, where L is the closest surface-to-surface separation of the spheres so that their centers have a distance

$d = R_2 - R_1 - L > 0$. For two concentric spheres, one has $Z = R_1/R_2$.

Drude model.—In general, a Drude metal has to be described within electromagnetic scattering theory. However, in the classical limit, it can be shown that only the electric or TM modes contribute to the Casimir energy [18]. We shall use this observation below to map the interaction of Drude metals to a scalar field problem that is similar to the one with Dirichlet conditions. To simplify notations, we focus here on the sphere-plate geometry (with the sphere described by $\mu = \mu_1 > 0$). When the operator $\hat{N}$ that yields the classical energy for the Drude model is expressed in a *spherical* wave basis with C as the origin (see Fig. 1), it has the form [18]

$$\hat{N} = \left[\left[\frac{(l + l')!}{(l + m)!(l' - m)!}(Z + Z^{-1})^{-l - l' - 1}\delta_{mm'}\right]\right], \quad (4)$$

where we used that $R/(L + R) = 2/(Z + 1/Z)$ for a plate and a sphere of radius R. The same matrix is obtained for a scalar field with Dirichlet conditions but with one important difference: While in the Dirichlet case monopoles with $l = 0$ are included, in the Drude case monopoles are excluded, since there are no corresponding spherical vector waves. Physically, this feature is a consequence of the fact that an *isolated* compact object has zero total charge, i.e., no monopole. The next important observation is that a given spherical multipole of order (l, m) decomposes into infinitely many bispherical multipoles (l', m) with m conserved, and, hence, in the bispherical basis the difference between Dirichlet and Drude conditions is more complex. Since monopoles $l = 0$ occur only in the sector for $m = 0$, it is sufficient to reconsider the part of the energy coming from $m = 0$ modes and to transfer the Dirichlet energy unchanged for $m \neq 0$, giving the classical Casimir energy for the Drude case

$$E^{(\mathrm{Dr})} = E_{m=0}^{(\mathrm{Dr})} + \frac{k_B T}{2}\sum_{l=1}^{\infty} 2l \ln(1 - Z^{2l+1}), \quad (5)$$

where $E_{m=0}^{(\mathrm{Dr})}$ is given by Eq. (2) with $\hat{N}$ from Eq. (4) with $m = 0$ and restricted to $l, l' \geq 1$, the equivalence class of which we denote by $\hat{N}_0$ in the following. To compute the determinant in Eq. (2), we perform two transformations on the matrix that represents $\hat{N}_0$: (i) a translation from the spherical wave basis S_C with origin C to a spherical wave basis S_F with origin F—see Fig. 1; (ii) a conversion from the basis S_F to the *bispherical* wave basis BS_F with the same origin F. The translation (i) over the distance ZR corresponds to a similarity transformation with a matrix which has nonzero elements only for $l \geq l' \geq 1$, given by

$$\mathcal{V}_{ll'}^{S_C \to S_F}(ZR) = \frac{(ZR)^{l-l'}}{(l - l')!}\frac{l!}{l'!}\sqrt{\frac{2l' + 1}{2l + 1}}.$$

The inverse of this matrix is obtained by $ZR \to -ZR$. After this transformation, we can write $\hat{N}_0$ as

$$\hat{N}_0 = \left[\left[\begin{array}{ll} -(-1)^l Z^{l+l'+1} & \text{if } l > l' \\ \left(\frac{l'!}{l!(l'-l)!} - (-1)^l\right)Z^{l+l'+1} & \text{if } l \leq l' \end{array}\right]\right]. \quad (6)$$

The conversion (ii) corresponds to a similarity transformation with a matrix which has nonzero elements only for $l' \geq l \geq 1$, given by

$$\mathcal{V}_{ll'}^{S_F \to BS_F} = \frac{(-1)^l}{\sqrt{2l + 1}}\left[R\left(\frac{1}{Z} - Z\right)\right]^{l+1/2}\frac{l'!}{l!(l' - l)!}.$$

With this transformation, the equivalence class can be expressed as

$$\hat{N}_0 = [[Z^{2l'+1}(\delta_{ll'} + (1 - Z^2)(1 - Z^{2l'}))]]. \quad (7)$$

The latter expression allows for a direct computation of $\det(1 - \hat{N}_0)$. The first part in Eq. (7) is diagonal and can be easily factorized so that it yields a contribution $(k_B T/2) \times \sum_{l=1}^{\infty} \ln(1 - Z^{2l+1})$ to the energy $E_{m=0}^{(\mathrm{Dr})}$. The second part in Eq. (7) depends only the column index l' and hence is a matrix with equal rows. For a matrix A of this type, one has $\det(1 - A) = 1 - \mathrm{tr}\, A$. Hence the contribution of the second part in Eq. (7) to the energy is given by the trace of this part divided by $1 - Z^{2l+1}$ due to the factorization of the first part of $1 - \hat{N}_0$. Combining the two parts from Eq. (7) with Eq. (5), we obtain the following *exact* expression for the classical Casimir energy of a Drude sphere and plate:

$$E^{(\mathrm{Dr})} = \frac{k_B T}{2}\left\{\sum_{l=1}^{\infty}(2l + 1)\ln(1 - Z^{2l+1})\right.$$
$$\left. + \ln\left[1 - (1 - Z^2)\sum_{l=1}^{\infty}Z^{2l+1}\frac{1 - Z^{2l}}{1 - Z^{2l+1}}\right]\right\}. \quad (8)$$

Applying similar arguments as in the Dirichlet case, it can again be shown that $E^{(\mathrm{Dr})}$ is an analytic function for $|Z| < 1$. We note that the effect of eliminating monopole fluctuations from the energy for Dirichlet conditions is not only a sum starting at $l = 1$ in $E^{(\mathrm{D})}$ but the occurrence of a second logarithmic term. This is a consequence of the coupling of monopoles to higher-order multipoles.

Expressions at short distances.—With exact expressions for the Casimir energies in the Dirichlet and Drude model available, one can compute explicitly the interaction in the experimentally important limit of short distances $L \lesssim R$. Since this limit corresponds to Z close to unity, we compute the series in Eqs. (3) and (8), using the Abel-Plana formula [2]. We set $Z = \exp(-\mu)$ and expand for small μ, where $\mu = \ln(1 + \ell + \sqrt{\ell(2 + \ell)})$, $\ell = L/R$, in the sphere-plate geometry. For Dirichlet conditions, we obtain

$$E^{(D)} = \frac{k_B T}{2}\left[-\frac{\zeta(3)}{2}\frac{1}{\mu^2} + \frac{1}{12}\ln\mu + \frac{1}{8} - \gamma_0 - \frac{7}{2880}\mu^2 \right.$$
$$\left. -\frac{31}{725\,760}\mu^4 + \mathcal{O}(\mu^6) \right] \qquad (9)$$

with the constant $\gamma_0 = 0.174\,897$ that is given by an integral [22], and for the Drude model we get

$$E^{(Dr)} = E^{(D)} + \frac{k_B T}{2}\left[\ln(\gamma_1 - \ln\mu) + \frac{1}{6}\frac{-\gamma_2 + \ln\mu}{-\gamma_1 + \ln\mu}\mu^2 \right.$$
$$\left. -\frac{1}{180}\frac{\gamma_3 - \gamma_4\ln\mu + \ln^2\mu}{(-\gamma_1 + \ln\mu)^2}\mu^4 + \mathcal{O}(\mu^6) \right], \quad (10)$$

where the constants $\gamma_1 = 1.270\,362$, $\gamma_2 = 1.353\,69$, $\gamma_3 = 1.594\,09$, and $\gamma_4 = 2.511\,53$ are given by integrals that can be easily computed numerically. We used μ as the variable for the expansion, since it provides a very accurate result at even larger distances; see Fig. 2. A general feature of both the Dirichlet and Drude energies is their dependence on only $\ln\mu$ and *even* powers of μ. This shows that the energies depend only on $\ln\ell$ and *integer* powers of ℓ. When the *force* for the Dirichlet
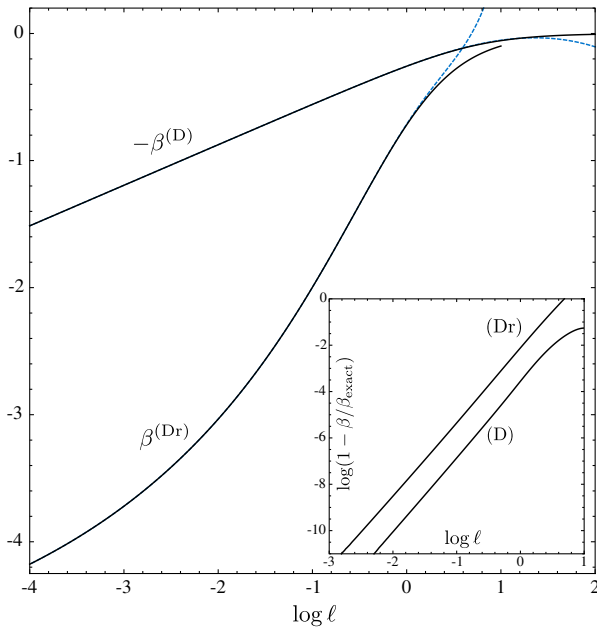

FIG. 2 (color online). Deviations from proximity force approximation in the sphere-plate geometry, measured by the function $\beta(\ell)$ defined by $E^{(D)/(Dr)} = -\zeta(3)k_B T/8[1/\ell + \beta^{(D)/(Dr)}(\ell)]$ with $\ell = L/R$. Shown are the exact results for Dirichlet conditions [Eq. (3)] and the Drude model [Eq. (8)] as solid lines and the short-distance expansions of Eqs. (9) and (10), as dashed lines as a function of the logarithm (with base 10) of ℓ. Note that $\beta^{(D)} > 0$ and $\beta^{(Dr)} < 0$ in the shown range. The inset shows the logarithmic relative difference between the exact result β_{exact} from Eqs. (3) and (8), and the β obtained from the short-distance expansions of Eqs. (9) and (10).

case is expanded in ℓ, it becomes a Laurent series starting with $1/\ell^2$; i.e., there are no logarithmic terms for the force. For the Drude model, there are logarithmic terms in the force, and the most convenient form to express the short-distance expansion appears to be the one in Eq. (10). The leading correction to the PFA is the same term $\sim \ln\mu$ in both models. However, at realistic distances, the double logarithmic term in Eq. (10) dominates, and therefore the two models show rather distinct behavior; see Fig. 2.

It is instructive to discuss the different behavior of the classical Casimir energies for Dirichlet and Drude boundary conditions. For large separations, the interactions display different asymptotic behaviors for the two boundary conditions. In the experimentally most relevant sphere-plate geometry Eq. (3) predicts the known $1/\ell$ falloff rate for the Dirichlet energy, while in the Drude case Eq. (8) gives the characteristic $1/\ell^3$ decay [23]. The slower decay rate in the Dirichlet case results from monopole contributions that are absent in the Drude case. While a Dirichlet scalar field is in general not expected to describe the Casimir interaction between metals, in the high-temperature limit the difference between the two universal interactions for Dirichlet and Drude conditions suggests an interesting physical interpretation. In fact, the high-temperature limit of the Casimir interaction provides the dominant contribution to the full quantum Casimir interaction at *finite* temperatures in the limit of sufficiently *large separations* [1,5]. To understand if the Dirichlet ($1/\ell$) or Drude ($1/\ell^3$) decay describes experiments with conductors at large distances, one must bear in mind that the conductors used in Casimir force measurements are always connected to a charge reservoir to compensate stray charges that might otherwise be present on the surfaces. In the static limit, the surface electric potential of such a conductor is constant (Dirichlet boundary conditions), and their total charge can fluctuate. In contrast, Drude metallic boundary conditions instead describe *ungrounded* charge-neutral conductors. From this we conclude that the quantum Casimir force between grounded conductors at finite temperatures should decay according to the Dirichlet case at large distances, since the energy is than dominated by the lowest Matsubara mode, i.e., the classical energy [1,5]. To discriminate between the asymptotic $1/\ell$ and $1/\ell^3$ decay, it is necessary to consider sphere-plate separations comparable to the sphere radius or larger. Interestingly, our short-distance expansions in the classical limit suggest a distinct behavior due to grounding at even shorter separations which, however, is certainly modified due to quantum corrections.

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