Topological Kondo Effect with Majorana Fermions

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The Kondo effect is a striking consequence of the coupling of itinerant electrons to a quantum spin with degenerate energy levels. While degeneracies are commonly thought to arise from symmetries or fine-tuning of parameters, the recent emergence of Majorana fermions has brought to the fore an entirely different possibility: a *topological degeneracy* that arises from the nonlocal character of Majorana fermions. Here we show that nonlocal quantum spins formed from these degrees of freedom give rise to a novel *topological Kondo effect*. This leads to a robust non-Fermi liquid behavior, known to be difficult to achieve in the conventional Kondo context. Focusing on mesoscopic superconductor devices, we predict several unique transport signatures of this Kondo effect, which would demonstrate the nonlocal quantum dynamics of Majorana fermions and validate their potential for topological quantum computation.

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Traditionally, the Kondo effect arises when conduction electrons couple to a confined region with a spin-degenerate ground state [1,2]. More intricate scenarios combining spin with other degeneracies can lead to exotic, non-Fermi liquid (NFL) behavior [3,4]. These degeneracies, however, require the fine-tuning of parameters, rendering such exotic physics quite fragile. Recent developments have shown that condensed matter systems can display another, much more robust, degeneracy called *topological degeneracy* [5]. This can arise from the appearance of localized Majorana fermions in certain superconductor structures [6,7].

The possibility of realizing Majorana fermions using superconductors has transformed an elusive notion of high-energy physics into a tangible excitation in electronic materials [8]. Several methods for creating them in a controlled manner have been proposed, building on such simple ingredients as *s*-wave superconductors and spin-orbit coupling [9,10]. This has led to the recent experimental observation [11] of conductance signatures indicating localized Majorana modes [12]. A key feature, not yet addressed experimentally, is that pairs of Majorana fermions can nonlocally encode zero-energy fermions, which span a multidimensional ground-state subspace. The degeneracy of the ground state is topological: it is ensured up to exponentially small corrections provided the Majorana fermions do not overlap. The resulting nonlocal zero-energy degrees of freedom form the topological qubits that underlie the proposed schemes for fault-tolerant quantum computation [5,6]. Finding the smoking gun signatures of their quantum dynamics is an urgent issue.

The Kondo effect provides a central paradigm leading to observable consequences of quantum dynamics within a degenerate ground state, but the possibility that the degeneracy has a topological origin has not been previously considered. We will show that topological degeneracy can be a source of novel exotic Kondo effects and NFL behavior that is highly robust. We predict that this *topological Kondo effect* leads to striking signatures in simple transport measurements on mesoscopic superconductor structures that support Majorana fermions. Such measurements can be used to give clear evidence for the quantum dynamics of nonlocal qubits, which form the basis for the proposed uses of Majorana fermions in fault-tolerant quantum computation.

We consider a setup consisting of a superconducting island supporting M_{tot} localized Majorana modes, M of which are coupled to spinless conduction electrons. The conduction electrons occupy M single-mode quantum wires (leads). As we explain below, achieving a Kondo effect requires $M_{\text{tot}} \geq 4$, $M \geq 3$. The simplest configuration with minimal M_{tot} and M is shown in Fig. 1. There can be several realizations, e.g., using superconducting heterostructures based on topological insulators [9], or semiconductor structures [10], as in the nanowire setup

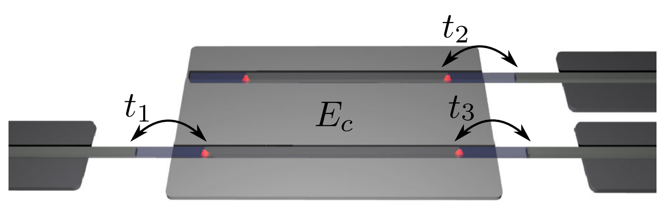

FIG. 1 (color online). Minimal setup for the topological Kondo effect. There are $M_{\text{tot}} = 4$ Majorana fermions (red dots), $M = 3$ of which are coupled to conduction electrons. The figure illustrates the realization based on semiconductor nanowires (horizontal bars). The wires are deposited on top of a superconductor (central rectangle). Nearby gates (not shown) put the central segment of the wires in a topological superconducting phase, while the adjacent segments are depleted, forming a tunnel barrier. The outermost segments host conduction electrons, and can be contacted to normal metal electrodes (outer rectangles).

in the experiment of Mourik *et al.*[11]. All realizations work in a regime where the electrons are effectively spinless (due to strong spin-orbit coupling [9] or a large Zeeman field [10]); the conditions ensuring this are naturally present also in the leads. We consider the superconductor to be of mesoscopic size, connected to the ground by a capacitor. In this regime, the charging energy E_c becomes important and can play a key role with regard to Majorana fermions [13]. It contributes to the Hamiltonian by a term $H_c(N) = E_c(N - \frac{q}{e})^2$, where N is the number of electrons on the island and q is a background charge determined by the voltage across the capacitor. The $M_{\rm tot}$ Majorana modes correspond to $M_{\rm tot}/2$ zero-energy fermionic modes ($M_{\rm tot}$ is always even). The parity of the total occupation number of these modes is tied to the parity of N. Therefore, in each N sector we have a $2^{M_{\rm tot}/2-1}$-fold ground-state degeneracy, which immediately shows why $M_{\rm tot} \geq 4$ is required. The typical energy gap Δ above the ground-state manifold is set by the superconducting proximity effect. For the realizations mentioned above, $\Delta \sim E_c \sim 0.5 - 1$ K is a reasonable estimate [13,14].

Working with temperatures and voltages $T, V \ll \Delta, E_c$ for weak lead-island coupling, the low energy physics is dominated by virtual transitions connecting the lowest energy ground-state manifold of charge eN to the ground states with $N \pm 1$ electrons [15]. This physics is captured by the effective Hamiltonian

$$H_{\rm eff} = \sum_{i\neq j} \lambda_{ij}^{+} \gamma_j \gamma_i \psi_i^\dagger \psi_j - \sum_i \lambda_{ii}^{-} \psi_i^\dagger \psi_i, \qquad (1)$$

where γ_i and ψ_i are Majorana and conduction electron operators (at the tunneling point i), respectively, and we have introduced the constants $\lambda_{ij}^{\pm} = (\frac{1}{U_+} \pm \frac{1}{U_-}) t_i t_j$, with $U_\pm = H_c(N \pm 1) - H_c(N)$ and tunneling amplitudes t_i (which can always be chosen positive). Equation (1) is obtained by a Schrieffer-Wolff transformation [1], implementing the leading-order perturbation theory in the lead-island couplings. The full Hamiltonian is $H = H_{\rm lead} + H_{\rm eff}$, where the first term is the Hamiltonian of the conduction electrons, which we assume to be noninteracting.

To explain how the Kondo problem emerges, let us focus on the first term in Eq. (1) and consider the setup of Fig. 1 with $M = 3$ coupled Majorana fermions. It is known (see, e.g., Ref. [6]) that the three γ_i realize the spin-1/2 Pauli matrices, $\boldsymbol{\sigma} = -(i/2)\boldsymbol{\gamma} \times \boldsymbol{\gamma}$. Coupling these to the three species in the leads suggests a Kondo problem of a spin-$\frac{1}{2}$ impurity with spin-1 conduction electrons [16,17]. Indeed, we have

$$\sum_{i\neq j} \lambda_{ij}^{+} \gamma_j \gamma_i \psi_i^\dagger \psi_j = \frac{1}{2} \sum_\alpha \lambda_\alpha \sigma_\alpha J_\alpha \qquad (2)$$

with $\lambda_\alpha = \sum_{ab} |\varepsilon_{\alpha ab}| \lambda_{ab}^{+}$, which is a Kondo term where the conduction electrons enter through the spin-1 object $J_\alpha = i\sum_{ab} \varepsilon_{\alpha ba} \psi_a^\dagger \psi_b$. Remarkably, the spin structure of J_α is distributed nonlocally to spatially separate leads; this will result in distinctive transport signatures.

The Kondo term is nontrivial if the impurity acts as a quantum spin, as opposed to a classical Ising variable. This requires coupling to at least two of the σ_α. This needs three γ_j, showing why $M = 3$ is the minimal case. The same σ_α are the Pauli matrices acting on the topological qubit [6]. The topological Kondo effect reveals the quantum spin nature of this object, thereby detecting the quantum qubit dynamics. A smoking gun signature of this is already clear: this Kondo effect should disappear if any one of the three leads is decoupled.

To see how the Kondo effect shows up, we begin with a renormalization group (RG) analysis of the minimal setup of Fig. 1. Our considerations also apply for $M_{\rm tot} > 4$, allowing for stray Majorana fermions not coupled to the leads. The presence of these modes is akin to the presence of uncoupled spins not participating in the Kondo effect.

In terms of bare parameters, $H_{\rm eff}$ enters as a weak perturbation. We obtain the RG flow in this weak coupling regime using the poor man's scaling procedure, giving

$$\frac{d\lambda_1}{dl} = \rho \lambda_2 \lambda_3, \qquad (3)$$

(and cyclic permutations of the indices 1, 2, 3) where ρ is the density of states of the leads at the Fermi energy. The couplings λ_{ij}^{-}, similar to the potential scattering terms in the Kondo context, do not renormalize. These are the usual weak coupling RG equations of the Kondo problem, but it should be kept in mind that λ_α now characterize nonlocal charge transfers between different leads. As $U_\pm, t_i > 0$, the bare Kondo coupling is antiferromagnetic. Typically, t_i will not have the same value, which translates into an exchange anisotropy in the Kondo language. Under Eq. (3) the couplings increase, while $\lambda_\alpha^2 - \lambda_\beta^2$ remain constant. The flow is therefore towards an isotropic coupling, $\lambda_\alpha/\lambda_\beta \to 1$. This conventional result in the Kondo context translates into something remarkable for our setup: a tendency towards a threefold symmetry with respect to relabeling the leads $j \to j + 1 (\mathrm{mod} 3)$. The overall behavior of the couplings is characterized by an inverse logarithmic growth,

$$\lambda_\alpha(\Lambda) \sim \frac{1}{\ln(\Lambda/T_{\rm K})}, \qquad (4)$$

where we have introduced the Kondo temperature $T_{\rm K}$ and the renormalized high-energy cutoff $\Lambda = \Lambda_0 e^{-l} \sim E_c e^{-l}$. Denoting by $\bar{\lambda}$ a typical bare value of the λ_α-s, one has $T_{\rm K} \sim E_c e^{-1/\rho\bar{\lambda}}$. The factors entering $T_{\rm K}$ are the same as for conventional Kondo systems, implying that considering Kondo temperatures anywhere in the range $0 < T_{\rm K} \lesssim 0.1$ K is reasonable [4].

Upon approaching $\Lambda \sim T_{\rm K}$, the couplings cease to be small, and the perturbative RG has to be replaced by a nonperturbative analysis. A powerful route is provided by the conformal field theory method of Affleck and Ludwig [18], which we applied to our problem [19]. (This complements Ref. [16], where the Kondo problem of spin-1 electrons was studied using Abelian bosonization assuming an axial symmetry, not present in our case.) We find that the flow is towards an intermediate coupling fixed point with NFL behavior, which is robust. In the vicinity of the fixed point, the scale dependence is due to dimension $4/3$ irrelevant operators $\mathcal{O}_\alpha$. The NFL behavior stems from these: $\mathcal{O}_\alpha$ cannot be constructed out of ordinary fermions, as the fermions can give only halfinteger dimensions. In addition, marginal operators, corresponding to the original λ_{ii}^- terms, will also be present, but their scale dependence (which is also through $\mathcal{O}_\alpha$) can be neglected. It is by itself remarkable that our setup, with simple noninteracting leads, allows for the appearance of NFL behavior without fine-tuning of the couplings. This is unlike conventional Kondo variants leading to NFL physics, where one has to fine-tune at least one coupling [3,4] or introduce leads that are themselves NFLs [20].

The weak coupling flow and the knowledge of the nature of the intermediate coupling fixed point can be applied to deduce the behavior of various experimentally relevant quantities. Given that the first signatures of localized Majorana modes were obtained by conductance measurements, we focus on the conductance G_{kl} between leads k and l. For simplicity, we work in the linear response regime and focus on the temperature dependence of G_{kl}. (The results also apply to the nonlinear differential conductance in the opposite $T \ll V$ case upon replacing T by V in the expressions.) The overall behavior is summarized in Fig. 2. Importantly, the smoking gun signature distinguishing Kondo and Ising cases is particularly clear: the G_{kl} we find is strikingly different from the small, temperature-independent conductance of the setup where the unmeasured lead is decoupled (i.e., the Ising case, realizing cotunneling transport [2] without the spin flips crucial to the Kondo effect).

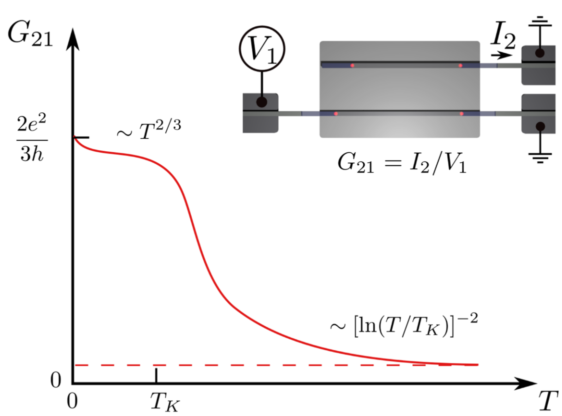

FIG. 2 (color online). Sketch of the predicted signatures of the topological Kondo effect showing the crossover between low and high temperature behaviors. The solid curve represents the temperature dependence of the offdiagonal conductance G_{21} in a setup shown in the Inset. For contrast, we also show (dashed line) G_{21} if the third lead is decoupled (Ising case). The striking difference between the two cases is a smoking gun signature of the topological Kondo effect.

We now discuss how we obtained the $T \gg T_\mathrm{K}$ and $T \ll T_\mathrm{K}$ asymptotics. In the $T \gg T_\mathrm{K}$ regime, one can apply a weak coupling argument: as the only term in H_eff that transfers charge between the leads is the Kondo coupling, the conductance is $G_{kl} \sim (\lambda_{kl}^+)^2$ to leading order in λ_{kl}^+. Combining this with Eq. (4), we find

$$G_{kl} \sim \frac{1}{\ln^2(T/T_\mathrm{K})}. \quad (5)$$

Observing such an inverse logarithmic increase would be a qualitative signature of the topological Kondo effect. Furthermore, we find $G_{ij}/G_{kl} \to 1$ as the temperature is lowered. This is a direct consequence of the flow to an isotropic Kondo coupling. Capturing this flow so directly via the conductance is a possibility that is absent in conventional Kondo settings, and it provides another qualitative signature of the topological Kondo effect.

At low temperatures $T \ll T_\mathrm{K}$, the inverse logarithmic increase crosses over to a power law convergence to the zero temperature limit. Suppressing the small, temperature-independent corrections due to the marginal operators, we find [19]

$$G_{kl}(T) = \frac{2e^2}{3h} + c_{kl} T^{2/3} \qquad (k \neq l), \quad (6)$$

where c_{kl} are nonuniversal coefficients [21]. This unusual T dependence directly signals the NFL physics—indeed, one can directly trace the $T^{2/3}$ law back to second-order corrections in $\mathcal{O}_\alpha$ [19]. Another noteworthy feature is that the diagonal conductances (which follow from G_{kl} through current conservation) approach $\frac{4e^2}{3h}$ as $T \to 0$. The fact that this value exceeds the conductance quantum indicates the presence of Andreev reflection processes, allowing for holes (not only electrons) to be backscattered. Note that these have a different origin than in conventional normal-superconductor systems where the superconductor absorbs a Cooper pair in the process. Indeed, the charging energy forbids this in our case. Instead, our system realizes a strongly correlated "Andreev reflection fixed point" [22], with the two electrons playing the role of Cooper pairs exiting through the leads. Detecting this enhanced conductance together with the $T^{2/3}$ dependence would be a clear signature of the NFL physics.

Before concluding, we briefly discuss the generalization of our results to $M > 3$. The M Majorana operators generate a Clifford algebra [8]. This implements the spinor representation of the Lie algebra of the orthogonal group $\mathrm{SO}(M)$ [23], with $i\gamma_j\gamma_k$ $(j < k)$ representing the jk-th $\mathrm{SO}(M)$ generator. These generalize σ_α in Eq. (2). The matrix $i\epsilon_{\alpha ab}$ generalizes to $A^{(jk)}_{ab} = i(\delta_{ja}\delta_{kb} - \delta_{ka}\delta_{jb})$, the jk-th SO(M) generator in the defining representation. For general $M < M_{\rm tot}$ we thus have an SO(M) Kondo problem, with a spinor impurity and conduction electrons in the defining representation. For $M = M_{\rm tot}$ the impurity is in a half-spinor representation depending on the parity of N. When this is faithful, lead electrons again furnish the defining SO(M) representation. To the best of our knowledge, Kondo problems of this type have not appeared in the literature so far. In particular, these problems are markedly different from the descriptions of the two-channel Kondo model related to orthogonal groups and/or Majorana fermions [24]. In addition to the apparent distinction that these works introduce Majorana fermions only for mathematical convenience, their models themselves are different from ours: they do not conserve charge [25], or have different group structure [24].

We end by outlining some features of the general $M < M_{\rm tot}$ case, assuming isotropic couplings $\lambda^{+}_{ij} = \lambda^{+}$, $\lambda^{-}_{ii} = \lambda_{-}$. (We expect that the results also hold for $M = M_{\rm tot}$ with faithful half-spinor representations.) The scaling (3) generalizes to

$$\frac{d\lambda^{+}}{dl} = 2\rho(M-2)(\lambda^{+})^2, \tag{7}$$

while λ^{-} does not renormalize. This again implies an inverse logarithmic growth of λ^{+} and the corresponding inverse log-square temperature dependence of the weak coupling conductance. At low temperatures, we expect NFL behavior, with a convergence to $G_{kl} = \frac{2e^2}{Mh}$ for $k \neq l$, obtained by generalizing the results [22] for the Andreev reflection fixed point.

In summary, we have shown that the topological degeneracy of Majorana fermions can lead to a new class of *topological* Kondo effects. These effects are not only novel from a mathematical perspective, but also have important and striking physical consequences for realistic experimental systems. We have established the detailed properties for the simplest case (with $M = 3$ leads coupled to Majorana fermions), in which topological degeneracy gives rise to a dynamical nonlocal quantum spin. We have shown that this leads to a NFL behavior that is robust to perturbations, in contrast to the conventional Kondo context, where such behavior is known to be unstable. The resulting nontrivial power law dependences and the enhanced conductance due to strong correlations are all distinctive qualitative features, which come with a smoking gun signature: they can be switched off at will by decoupling any one of the three leads. The physics we describe can readily be explored in experiments on mesoscopic devices based on superconducting structures using available technology. These studies would provide a clear test of the expected nonlocal quantum dynamics of Majorana fermions: such a measurement would be a crucial step towards establishing the Majorana architecture for fault-tolerant quantum computation.

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