Modification of Late-Time Phase Structure by Quantum Quenches

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The consequences of the sudden change in the coupling constants (quenches) on the phase structure of the theory at late times are explored. We study in detail the three-dimensional ϕ^6 model in the large-N limit and show that the ϕ^6 coupling enjoys a widened range of stability compared to the static scenario. Moreover, a new massive phase emerges, which for sufficiently large coupling becomes the dominant vacuum. We argue that these novel phenomena cannot be described by a simple thermalization effect or the emergence of a single effective temperature.

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Unlocking the problem of out-of-equilibrium dynamics of a quantum coherent system is one of the fundamental questions in quantum physics. This is particularly true in the context of quantum field theories, where many important questions have so far been addressed mainly in a static scenario, such as the renormalization group flow, the phase structure of vacua, and critical points. There are also interesting questions that spring directly from a nonequilibrium system, such as the mechanisms of relaxation, the time scale over which this occurs, and the existence of an effective description at late times. These problems, given their fundamental role in field theories, naturally appear in many places. For example, it is not surprising that nonequilibrium dynamics are important in cosmology, in the evolution of the early universe. The Relativistic Heavy Ion Collider experiments, involving the relaxation of the quark-gluon plasma, is another such example. Dynamical systems appear frequently in the context of condensed matter physics. Recently, the study has been rendered particularly pertinent experimentally due to new advances in the control of cold atomic gases [1–4]. For the first time, we are able to observe minute details of the evolution of a system that retains its quantum coherence for sufficiently long periods of time. One class of situations that has been subjected to intensive studies is called the quantum quench, in which a particular external field, or parameter, of the system is changed abruptly. An example is a sudden change of the external magnetic field to which the atoms couple. These experiments have inspired a flurry of theoretical activities, most notably initiated by Calabrese and Cardy [5]. Previous works, however, have been concentrated on free field theories, one-dimensional interacting theories, and integrable models (see, e.g., Refs. [6–12]). Attempts to understand interacting theories in higher dimensions by considering the large-N limit of a ϕ^4 theory have been made in Ref. [13] (see also a related problem in Ref. [14]).

Previous studies of the quenches mostly concerned the relaxation of the system. One central issue is whether the system thermalizes and is therefore describable by an effective temperature at late times. It is, however, an open problem if thermalization occurs at all and, if it does not, which is shown to be the case in many integrable models and even in some interacting models (see, for example, Ref. [15] and references therein), whether there are convenient effective descriptions of such systems and observables or effective parameters that characterize their behavior. This leads us to the current investigation of the phase structure of some out-of-equilibrium states, which in the scenario concerned is prepared by a quantum quench. This should be contrasted with the usual notion of the phase structure of a given Hamiltonian, which is a property of its ground state. Here, we have to deal with a state that, while settling to some static equilibrium in the far future, does not resemble a thermal state, nor is it able to relax to the ground state because of its isolation and energy conservation after the quench. It is, therefore, only natural to consider fluctuations about such a special state as opposed to the ground state, and determine its corresponding phase structure. We demonstrate that this phase structure differs significantly from that of the ground state even at late times as the system approaches equilibrium again.

In particular, we explore the $g_6\phi^6$ theory at the tricritical point, i.e., when all dimensionful couplings immediately after the quench are tuned to zero. What is special about this model is that it was shown to possess an ultraviolet fixed point $g_6 = 192$, using the $1/N$ expansion [16,17]. This fixed point, however, lies in the instability region of the model where nonperturbative effects dominate [18]. (See also Ref. [19] for a recent analysis of the β function in the case of three-dimensional Chern-Simons theories coupled to a scalar field in the fundamental representation.) The latter implies that the theory is always driven into the unstable region and, therefore, does not make physical sense. We revisit this theory in the context of quantum quenches. To that end, we employ the methods introduced in Ref. [13], where the effect of a quench is incorporated as a boundary condition on the fields. Assuming that the system does settle down, we then self-consistently compute the effective potential that defines the phase structure of the theory at late times. A corresponding phase diagram is obtained. Surprisingly, it is modified dramatically in comparison to the unquenched case. The region of stability is substantially widened such that the UV fixed point of the β function now lies well within. Moreover, a new stable minimum in the effective potential emerges when the coupling constant exceeds the upper bound of the stability range in the static theory. The new vacuum starts life as a metastable phase, but then becomes dominant for sufficiently large values of the coupling. In particular, the effective mass of the new phase increases as the coupling increases and, eventually, diverges when the coupling hits the boundary of a newly established range of stability.

Quenching the scalar model.—The scalar $O(N)$ vector model consists of an N-component scalar field ϕ. For simplicity we assume that initially the theory is free and the system is prepared in the ground state of a free Hamiltonian $|\Psi_0\rangle$. At $t = 0$ the marginal ϕ^6 interaction as well as the relevant ϕ^4 interaction are instantaneously switched on, and at the same instant the bare mass parameter of the field jumps from μ_0 to μ. The action of the system after the quench is given by

$$S(\phi) = \frac{1}{2}\int d^3x\bigg[\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2 - \frac{g_4}{2N}(\phi^2)^2 - \frac{g_6}{3N^2}(\phi^2)^3\bigg].$$

Since parameters of the theory are changed abruptly rather than adiabatically, one needs to resort to the well-known Keldysh-Schwinger, or in-in, formalism for non-equilibrium quantum systems. In this formalism, the integration over the time coordinate t in the path integral starts from some initial time t_i, extends to some final time t_f, and then goes back to t_i. Correlation functions are path ordered. In this approach, one needs to impose boundary conditions at $t = t_i$. In our case, we require that the initial state at $t_i = 0$ be given by $|\Psi_0\rangle$. The expectation value of an arbitrary operator $\hat{\mathcal{O}}(t)$ is thus given by

$$\langle\Psi_0|\hat{\mathcal{O}}(t)|\Psi_0\rangle = \int_{\text{CTP}} D\phi\,\hat{\mathcal{O}}(t)e^{iS(\phi)}, \quad (1)$$

where, for brevity, we used the following notation to designate the closed-time-path integral measure

$$\int_{\text{CTP}} D\phi = \int D\phi_i\Psi_0(\phi_i)\int D\tilde{\phi}_i\Psi_0^*(\tilde{\phi}_i)\int_{\phi_i}^{\tilde{\phi}_i} D\phi, \quad (2)$$

where ϕ_i and $\tilde{\phi}_i$ denote the values of the scalar field ϕ at the end points of the time contour, whereas $\Psi_0(\phi_i) = \langle\phi_i|\Psi_0\rangle$.

Introducing the following identity into the path integral [20]

$$I \sim \int_{\text{CTP}} D\rho\,\delta(\phi^2 - N\rho) \sim \int_{\text{CTP}} D\rho D\lambda e^{-i/2\int d^3x\lambda(\phi^2 - N\rho)}, \quad (3)$$

yields

$$\langle\Psi_0|\hat{\mathcal{O}}(t)|\Psi_0\rangle = \int_{\text{CTP}} D\phi\int D\rho D\lambda\hat{\mathcal{O}}(t)e^{iS(\phi,\rho,\lambda)}, \quad (4)$$

where

$$S(\phi,\rho,\lambda) = \frac{1}{2}\int d^3x\bigg[\partial_\mu\phi\partial^\mu\phi - (\mu^2+\lambda)\phi^2 - N\frac{g_4\rho^2}{2} - N\frac{g_6\rho^3}{3} + N\rho\lambda\bigg]. \quad (5)$$

Now performing the Gaussian integral over ϕ leads to

$$\langle\Psi_0|\hat{\mathcal{O}}(t)|\Psi_0\rangle = \int_{\text{CTP}} D\rho D\lambda\hat{\mathcal{O}}(t)e^{iNS_{\text{eff}}(\rho,\lambda)}, \quad (6)$$

with

$$S_{\text{eff}}(\rho,\lambda) = \int d^3x\bigg[\frac{\lambda\rho}{2} - \frac{g_4}{4}\rho^2 - \frac{g_6}{6}\rho^3\bigg] + \frac{i}{2}\operatorname{Tr}\log(\partial^2 + \mu^2 + \lambda). \quad (7)$$

The first thing to note about the above expression is that boundary conditions are now encoded in the functional trace. Secondly, this trace explicitly depends on the integration parameter λ, and this in turn renders evaluation of the remaining path integral very difficult.

However, in the limit when N is large while g_4 and g_6 are fixed, the right-hand side of Eq. (6) is dominated by the field configurations that minimize the right-hand side of Eq. (7). The effective mass can thus be evaluated as follows:

$$m_\phi^2 = \mu^2 + g_4\bar{\rho} + g_6\bar{\rho}^2, \qquad \bar{\rho} = \int\frac{d^2p}{(2\pi)^2}\tilde{G}_\phi(t,t;p), \quad (8)$$

where $m_\phi^2 = \mu^2 + \bar{\lambda}$ is the effective mass of the scalar field and $\tilde{G}_\phi(t_1,t_2;p)$ is the full momentum space two-point correlation function of the scalar field to leading order in $1/N$. Fields evaluated at the saddle point are denoted by a bar.

Note that $\tilde{G}_\phi(t_1,t_2;p)$ depends on the effective mass m_ϕ^2, and therefore it is difficult to solve Eq. (8) in full generality. Hence, in what follows we use the approximation proposed in Ref. [13]. In particular, we assume that m_ϕ tends to a stationary value m_ϕ^* and that this happens fast enough to be approximated by a jump. Then the two-point correlation function $\tilde{G}_\phi(t_1,t_2;p)$ is approximately the same as the propagator in the massive free field theory in which the physical mass is instantaneously changed from μ_0 to m_ϕ^*, i.e.,

$$\tilde{G}_\phi(t_1,t_2;p) \simeq G_\phi(t_1,t_2;p;\mu_0,m_\phi^*), \quad (9)$$

where [13]

$$G_\phi(t_1, t_2; p; \mu_0, m^*_\phi) = \frac{(\omega^*_p - \omega_{0p})^2}{4\omega^{*2}_p \omega_{0p}} \cos\omega^*_p(t_1 - t_2) + \frac{\omega^{*2}_p - \omega^2_{0p}}{4\omega^{*2}_p \omega_{0p}} \cos\omega^*_p(t_1 + t_2) + \frac{1}{2\omega^*_p} e^{-i\omega^*_p|t_1 - t_2|}, \quad (10)$$

with $\omega^*_p = \sqrt{\vec{p}^2 + m^{*2}_\phi}$ and $\omega_{0p} = \sqrt{\vec{p}^2 + \mu^2_0}$. The second term on the right-hand side is the only one that breaks time translation invariance. However, using stationary phase approximation, it can be shown that its contribution to $\bar{\rho}$ vanishes as $t^{-1/2}$ for $t_1 = t_2 = t \to \infty$. Note that in Eq. (8), $\bar{\rho}$ is divergent. We take a sharp cut off Λ to regulate the divergent integral over the momentum, and further absorb Λ in the bare couplings by the following renormalization scheme: $\mu^2_R = \mu^2 + g_4\Lambda/(4\pi) + g_6\Lambda^2/(16\pi^2)$, and $g^R_4 = g_4 + g_6\Lambda/(2\pi)$. This yields

$$m^{*2}_\phi = \mu^2_R - \frac{g^R_4}{4\pi}\left[\mu_0 + \frac{1}{2}\sqrt{m^{*2}_\phi - \mu^2_0}\arccos(\mu_0/m^*_\phi)\right] + \frac{g_6}{16\pi^2}\left[\mu_0 + \frac{1}{2}\sqrt{m^{*2}_\phi - \mu^2_0}\arccos(\mu_0/m^*_\phi)\right]^2. \quad (11)$$

Solutions of this equation describe the stationary points of the effective potential.

Phase structure of the model.—To analyze the admissible phases of the model let us derive the effective potential of the theory as $t \to \infty$. From Eq. (7), we get, up to a $\bar{\lambda}$-independent constant,

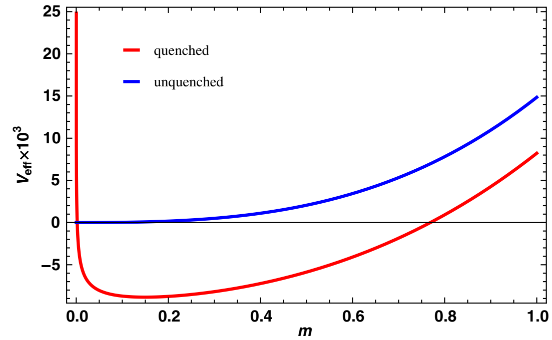

FIG. 1 (color online). Red line (bottom): Effective potential [Eq. (13)] as a function of m for $g_6 = \mu_R = 0$ and $\tilde{g}^R_4 = 1$. Blue line (top): Effective potential of the ϕ^4 theory; i.e., $g_6 = 0$ in the absence of quench when $\mu_R = 0$ and $\tilde{g}^R_4 = 1$.

$$V_{\rm eff}(\bar{\rho}, m^{*2}_\phi) = \frac{\mu^2}{2}\bar{\rho} + \frac{g_4}{4}\bar{\rho}^2 + \frac{g_6}{6}\bar{\rho}^3 - \frac{m^{*2}_\phi\bar{\rho}}{2} + \frac{1}{2}\int_0^{m^{*2}_\phi} dm^2 \int^\Lambda \frac{d^2p}{(2\pi)^2} G_\phi(t, t; p; \mu_0, m). \quad (12)$$

Varying this effective potential with respect to $\bar{\rho}$ and m^{*2}_ϕ correctly reproduces the saddle point Eq. (8). Note that ρ is not a dynamical field since it enters only algebraically into the action [Eq. (7)]. Hence, we eliminate it from the effective potential using the second part of Eq. (8). Replacing the couplings by renormalized ones and further rescaling them by μ_0 yields

$$\tilde{V}_{\rm eff}(m^2) = -\frac{\tilde{\mu}^2_R}{8\pi}\left[1 + \frac{1}{2}\sqrt{m^2 - 1}\arccos(1/\sqrt{m^2})\right] + \frac{\tilde{g}^R_4}{4(4\pi)^2}\left[1 + \frac{1}{2}\sqrt{m^2 - 1}\arccos(1/\sqrt{m^2})\right]^2 - \frac{g_6}{6(4\pi)^3}\left[1 + \frac{1}{2}\sqrt{m^2 - 1}\arccos(1/\sqrt{m^2})\right]^3 + \frac{(m^2 + 2)\sqrt{m^2 - 1}\arccos(1/\sqrt{m^2}) + m^2 - \log m^2}{48\pi}, \quad (13)$$

where $\tilde{V}_{\rm eff}$, $\tilde{\mu}_R$, $\tilde{g}^R_4$, and m^2 denote, respectively, the rescaled dimensionless effective potential, the dimensionless renormalized couplings μ_R and g^R_4, and asymptotic mass. Again, varying Eq. (13) with respect to m^2 recovers Eq. (11).

Let us briefly discuss the case where both g_6 and μ_R are zero. This case was considered in Ref. [13]. The characteristic shape of the effective potential of the quenched theory in this case is shown in Fig. 1 (red line), and as pointed out in Ref. [13] a finite mass always emerges at late times in the presence of interactions. This should be contrasted with the presence of a global minimum at $m = 0$ in the unquenched theory as shown in Fig. 1 (blue line). This case already illustrates the main point that we wish to make, namely, that the shape of the effective potential at late times depends on the quench, an event that occurred in the far past. More interesting and spectacular, however, is the case in which the theory sits at the tricritical point, i.e., when $\mu_R = g^R_4 = 0$. Expanding the effective potential then for large and small values of m^2 yields

$$\tilde{V}_{\rm eff}(m) = \frac{m^3}{96}\left(1 - \frac{g_6}{g_c}\right) + \mathcal{O}(m) \quad \text{if } m \gg 1,$$
$$\tilde{V}_{\rm eff}(m) = -\frac{g_6}{g_c}\frac{\log^3 m}{12\pi^3} + \mathcal{O}(1) \quad \text{if } m \ll 1, \quad (14)$$

where $g_c = 256$ corresponds to a critical value beyond which the potential is unbounded from below, and thus the theory is unstable.

It is remarkable that g_c is larger than the corresponding value in the unquenched case [18]. There, the region of stability is bounded by $0 \le g_6 \le (4\pi)^2 \equiv g^*_6$ (the lower bound $g_6 \geq 0$ is necessary to avoid instability in the direction of $O(N)$ broken phase [18]). In particular, previous studies [16–18] have spelled disaster for the theory in the ultraviolet limit: the β function drives the system into a UV fixed point $g_6 = 192$, which lies beyond the region of stability. In contrast, our results indicate that there is a way to circumvent the above conclusion by a quench in the parameters of the system.

It is instructive to contrast the phase diagrams with those of the unquenched case. There is only one admissible conformal phase if the coupling constant satisfies $0 \leq g_6 < (4\pi)^2$ and the system is not quenched. Quenching introduces a scale μ_0, and even though right after the quench the theory enjoys conformal invariance, μ_0 is inherent to the state and thus sets a scale for all dimensionful observables. In particular, the system resides in a light massive phase which scales with μ_0 (see the solid blue graph at the top of Fig. 2).

If the coupling constant is tuned to a special value g_6^*, the unquenched potential becomes flat, and thus a continuum of massive solutions emerges. This continuum is associated with spontaneous breaking of the scale invariance, and it coexists with the massless phase. In the quenched case, the energy scale μ_0 singles out a unique vacuum out of this continuum. When $g_6 = g_6^*$, this vacuum manifests itself as a point of inflection in the effective potential. For $g_6^* < g_6 < g_c$ as $t \to \infty$ and the system comes to equilibrium, we find two admissible vacua, shown in Fig. 2. The heavier phase is only metastable unless the coupling constant is sufficiently close to g_c (see the dashed green and solid black graphs at the bottom of Fig. 2). In the unquenched case, all phases become unstable for $g_6 > g_6^*$, and the system rolls down to infinity [21].

At the tricritical point, this model exhibits conformal invariance. Therefore, the energy-momentum tensor satisfies the operator equation $T^\mu_\mu = 0$, and the expectation value of its trace in the state that emerges after the quench must vanish. Indeed,

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{\eta_{\mu\nu}}{2}\left[(\partial\phi)^2 - \frac{g_6}{3N^2}\phi^6\right] - \frac{1}{8}(\partial^2_{\mu\nu} - \eta_{\mu\nu}\partial^2)\phi^2. \tag{15}$$

The last term does not contribute to the expectation value since $N\bar{\rho} = \langle\phi^2\rangle$ is constant as $t \to \infty$. Furthermore, from Eq. (10) it follows that asymptotically $\langle(\partial\phi)^2\rangle = N m_\phi^{*2}\bar{\rho}$. Hence,

$$\langle T^\mu_\mu\rangle = -\frac{1}{2}\langle(\partial\phi)^2\rangle + \frac{g_6}{2N^2}\langle(\phi^2)^3\rangle = \frac{N\bar{\rho}}{2}(g_6\bar{\rho}^2 - m_\phi^{*2}) = 0, \tag{16}$$

where the last equality follows from Eq. (8).

To conclude, by studying the late-time phase structure of the ϕ^6 theory after a quantum quench, we have demonstrated the following: a dramatic event that occurred in the far past can have significant effects even in the far future. Not entirely unexpectedly, we find that in the large-N limit the late-time physics cannot be described by simple thermalization with a single effective temperature, which has been noted in many integrable models. The temperature obtained by inspecting the fluctuation of the scalar field does not agree with that resulting from matching the expectation value of the stress tensor with its thermal counterpart [22]. Furthermore, generically the free energy is manifestly different from the effective potential [Eq. (13)]; see Fig. 3. The quantum quench modified significantly the phase structure long after the system has relaxed and settled into an equilibrium state. We believe this is a generic feature of quantum quenches and is not specific to the model that we have studied. In passing, we note also significant modifications in a supersymmetric version of this model: contrary to the current model, where new stable phases are created, we found instability generated by the quench [23]. This apparent dependence of the phase structure on past events may have implications in other areas of

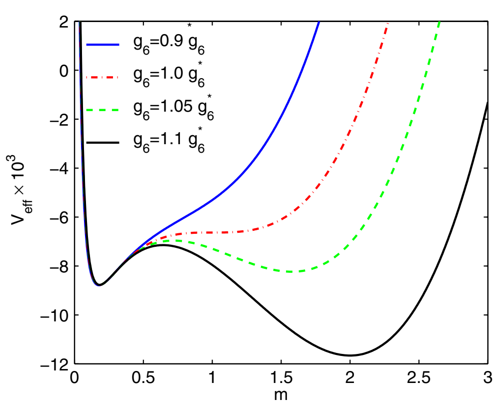

FIG. 2 (color online). Effective potential [Eq. (13)] as a function of m at the tricritical point $\mu_R = g_4^R = 0$ in the vicinity of $g_6^* = (4\pi)^2$.

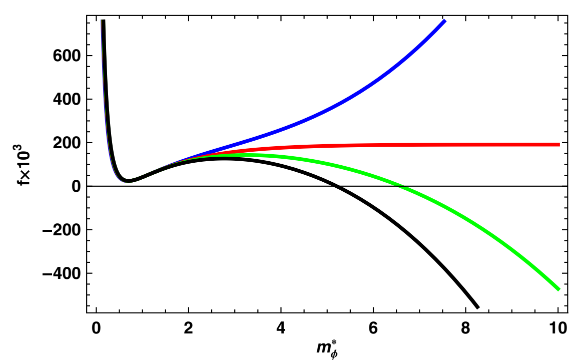

FIG. 3 (color online). The plot of the free energy density f for various values of the coupling constant g_6 as a function of effective mass m_ϕ^*. The inverse temperature is set to $\beta = 1$, and $g_6 = 0.9g_6^*$, g_6^*, $1.05g_6^*$, $1.1g_6^*$ from the top to the bottom graph, respectively.

physics, e.g., in cosmology. It is therefore important to determine, and perhaps classify, different time-dependent changes in a generic theory that could potentially lead to drastic modification of late-time physics. In this work, we have made extensive use of techniques developed in Ref. [13], where it is implicitly assumed that the system relaxes ultimately to an equilibrium. The authors of Ref. [13] support their claim by extensive numerical computations and show that this assumption works very well quantitatively in ϕ^4 theories in arbitrary dimensions. In this work, we extrapolate their assumption to study the ϕ^6 theory at the tricritical point. Further numerical checks of the current model and more general ones are under way and will appear elsewhere [23].

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