Zeno Dynamics, Indistinguishability of State, and Entanglement

Augusto Smerzi
QSTAR, INO-CNR and LENS, Largo Enrico Fermi 2, 50125 Firenze, Italy
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According to the quantum Zeno effect (QZ), frequent observations of a system can dramatically slow down its dynamical evolution. We show that the QZ is a physical consequence of the statistical indistinguishability of neighboring quantum states. The time scale of the problem is expressed in terms of the Fisher information and we demonstrate that the Zeno dynamics of particle entangled states might require quite smaller measurement intervals than classically correlated states. We propose an interferometric experiment to test the prediction.

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Introduction.—By watching a quantum system we can freeze its dynamical evolution [1]. This vindicates one of the classical Zeno paradoxes arguing that at every instant of time a flying arrow is motionless since it occupies a space equal to its own length. An infinite sum of zero displacements is still zero and therefore the arrow does not move [2]. In the quantum world, Zeno's prediction would be confirmed experimentally if we try to measure the displacements at each instant of time. The nonunitary measurements (or unitary strong couplings with external systems [3]) will repeatedly bring back the arrow to its initial position. The central result of this Letter is to show that this happens if, and only if, the different quantum states of the flying arrow are statistically indistinguishable in terms of the measurement results.

The quantum Zeno effect (QZ) has raised and continues to gather widespread interest mainly for two reasons: for its foundational implications about the nature of a "quantum measurement" [4] and for its technological applications in quantum information. Indeed, the QZ can be exploited, for instance, to create decoherence-free regions for quantum computation protocols [5–7]. Various aspects of the QZ have been experimentally demonstrated with ions [8], polarized photons [9], cold atoms [10], and dilute Bose-Einstein condensed gases [11]. The paradoxical (or, at least, surprising) nature of the problem, however, had sometimes obscured its physical significance [4,12].

The basics of the quantum Zeno effect can be illustrated with a Hamiltonian $\mathcal{H}$ driving a pure state $|\psi_0\rangle$ to an orthogonal state $|\psi(t_f)\rangle = e^{-i\mathcal{H}t_f}|\psi_0\rangle$ at time t_f. The survival probability, namely the probability to find the evolved state in its initial configuration, is $P(t_f) = |\langle\psi_0|\psi(t_f)\rangle|^2 = 0$. Suppose now that during the dynamics the system is probed m times at intervals $\tau = t_f/m$ with projective measurements $\Pi = |\psi_0\rangle\langle\psi_0|$. The survival probability becomes $P(t_f) = |\langle\psi_0|e^{-i\mathcal{H}\tau}\times \Pi e^{-i\mathcal{H}\tau}...\Pi e^{-i\mathcal{H}\tau}|\psi_0\rangle|^2 = |\langle\psi_0|\psi(\tau)\rangle|^{2m}$. For small τ we can expand $e^{-i\mathcal{H}\tau}$ up to second order and have:

$$P(t_f) = |\langle\psi_0|\psi(\tau)\rangle|^{2m} \simeq 1 - m\Delta^2\mathcal{H}\tau^2, \tag{1}$$

where $\Delta^2\mathcal{H}$ is the variance of the energy. The initial state does not evolve with time and remains frozen to its initial configuration with survival probability $P(t_f) \simeq 1$ when $m\Delta^2\mathcal{H}\tau^2 = \Delta^2\mathcal{H}t_f^2/m \ll 1$, which can be always satisfied with a sufficiently large number of measurements m (small measurement intervals τ) [13].

The quantum Zeno effect does not necessarily freeze the system. If the measurements project on a multidimensional subspace, the state can freely evolve within it [14]. Let's consider a unitary dynamics $e^{-i\mathcal{H}t}\rho_0 e^{i\mathcal{H}t}$ in the Hilbert space H and projections Π not commuting with the Hamiltonian. The initial state ρ_0 is taken in the "Zeno subspace" $H_\Pi = \Pi H$ so that $\rho_0 = \Pi\rho_0\Pi$ and $\mathrm{Tr}[\rho_0\Pi] = 1$. A sequence of m observations can repeatedly bring the system back inside H_Π with survival probability

$$P(t) = \mathrm{Tr}[V(\tau)^m\rho_0 V^\dagger(\tau)^m], \tag{2}$$

and final state

$$\rho(t) = \frac{V(\tau)^m\rho_0 V^\dagger(\tau)^m}{P(t)}, \tag{3}$$

where $V(\tau)^m = (\Pi e^{-i\mathcal{H}\tau}\Pi)^m$ and $\tau = t/m$. In the limit $m \to \infty$, Eq. (3) evolves unitarily inside the Zeno subspace, $e^{-i\mathcal{H}_\Pi t}\rho_0 e^{i\mathcal{H}_\Pi t}$, with $\mathcal{H}_\Pi = \Pi\mathcal{H}\Pi$ and $P(t) \to 1$ [14]. Such "Zeno dynamics" is the most general manifestation of the quantum Zeno effect.

In the following we demonstrate that the Zeno dynamics is a physical consequence of statistical indistinguishability. The small parameter of the theory is written in terms of the Fisher information, which provides a measure of distinguishability in the space of density operators. We show that particle entangled states might require quite smaller measurement intervals than classically correlated states and we finally propose an interferometric experiment to test the prediction.

Zeno dynamics, distinguishability of quantum states, and Fisher information.—We first shortly review the concept of statistical distinguishability. Let's consider a path parametrized by τ through the space of quantum states (τ can be, for instance, a phase shift due to the interaction with some external perturbation or an elapsed time). Wootters introduced the concept of statistical length (or distinguishability) [15] as the number of states along the path which can be physically discriminated with a large number m of measurements. In the limit $m \gg 1$, the smallest path interval $\tau_d = \tau_1 - \tau_0$ such that two states $\rho(\tau_0)$ and $\rho(\tau_1)$ are statistically distinguishable is [15]:

$$\tau_d = \frac{2}{\sqrt{m}\sqrt{F(\tau_0)}}. \tag{4}$$

The leading role in the theory is played by the Fisher information:

$$F(\tau) = \int d\eta \frac{1}{\mathcal{P}(\eta|\tau)}\left(\frac{d\mathcal{P}(\eta|\tau)}{d\tau}\right)^2, \tag{5}$$

where $\mathcal{P}(\eta|\tau) = \mathrm{Tr}[M(\eta)\rho(\tau)]$ is the likelihood, i.e., the conditional probability to obtain from the measurement a value η for a given τ and $M(\eta)$ is a generic observable with $\int d\eta M(\eta) = I$ (identity). We will further discuss Eqs. (4) and (5) at the end of this section.

We now show that the Fisher information plays a central role in the quantum Zeno dynamics. The first step is to expand the survival probability Eq. (2) for small time intervals τ [16]:

$$P(t) \simeq 1 - m\Delta^2 H\tau^2, \tag{6}$$

where $H = \mathcal{H} - \mathcal{H}_\Pi$. The initial state ρ_0 evolves in the Zeno subspace when $m\Delta^2 H\tau^2 \ll 1$ [17].

The projective measurements have only two possible outputs, which we call "yes" and "no," corresponding to the evolved state being projected inside the Zeno subspace with probability $\mathcal{P}(\text{yes}|\tau) = Tr[\Pi e^{-i\mathcal{H}\tau}\rho(t)e^{i\mathcal{H}\tau}\Pi]$ or gone with $\mathcal{P}(\text{no}|\tau) = 1 - \mathcal{P}(\text{yes}|\tau)$. After replacing these probabilities in Eq. (5) we have

$$F(\tau) = \left(\frac{\mathcal{P}(\text{yes}|\tau)}{\partial\tau}\right)^2 \frac{1}{\mathcal{P}(\text{yes}|\tau)[1 - \mathcal{P}(\text{yes}|\tau)]}. \tag{7}$$

Expanding in τ, $\mathcal{P}(\text{yes}|\tau) \simeq 1 - \Delta^2 H\tau^2$ and we obtain

$$F(\tau) = 4\Delta^2 H + O(\tau^4). \tag{8}$$

Notice that the Fisher information is independent from time $t = m\tau$, i.e., from the number of measurements m and from τ up to $O(\tau^4)$ [16]. By replacing (8) in (6) we can write the survival probability in terms of the Fisher information:

$$P(t) \simeq 1 - \frac{F}{4m}t^2 = 1 - \left(\frac{\tau}{\tau_{\mathrm{QZ}}}\right)^2. \tag{9}$$

Equation (9) quite generally describes the quantum Zeno dynamics from the case of simple projective measurements to more sophisticated "bang-bang" and continuous measurements. The ratio τ/τ_{QZ} is the small parameter of the theory, with the quantum Zeno time

$$\tau_{\mathrm{QZ}} = \frac{2}{\sqrt{mF}} = \frac{1}{\sqrt{m\Delta^2 H}}, \tag{10}$$

providing the natural time scale of the problem: QZ is created by measurements made at intervals smaller than τ_{QZ}. Moreover, by comparing (4) and (10), we have that

$$\tau_{\mathrm{QZ}} = \tau_d. \tag{11}$$

The Zeno time τ_{QZ} coincides with the largest interval τ_d such that the two states remain indistinguishable.

Equations (9)–(11) summarize the main result of this Letter. The quantum Zeno dynamics is the physical consequence of an interval τ between measurements small enough such that the set of evolved states $\{\rho_1 = e^{-i\mathcal{H}\tau}\rho(t)e^{i\mathcal{H}\tau}\}$ with $t = k\tau,\ k = 0, 1, \ldots m$ is statistically indistinguishable from the corresponding initial states $\{\rho(t)\}$ in terms of the m consecutive measurements done during the dynamics. For pure states, this concurs to an evolved state $|\psi(\tau)\rangle = e^{-i\mathcal{H}\tau}|\psi_0\rangle$ being statistically indistinguishable from the initial state $|\psi_0\rangle$ with m projective measurements. As a consequence of indistinguishability each projection steadily brings back the evolved state inside the Zeno subspace with survival probability $P(t) \to 1$ when $\tau \to 0$, $m \to \infty$, $t = m\tau$.

Before continuing, it is worth looking back at the Fisher information. Equations (4) and (5), are a general result of statistical analysis which holds in, both, quantum and classical frameworks. This is because $F(\tau)$ only depends on conditional probabilities which can naturally incorporate quantum fluctuations and low detection efficiencies, noise and decoherence [18]. However, the Zeno dynamics is a purely quantum effect and the results Eqs. (9)–(11) are obtained assuming a unitary evolution of a generic density operator. The analysis remains valid, in principle, also in the presence of an "environment" but with the crucial caveat to consider the full Hamiltonian of the system. Quite obviously, in the case of a "strong environment" the Zeno time scale might turn out to be so small that it makes the Zeno dynamics unaccessible. Extensions of the theory by considering more general positive operator valued measurements (POVM) [4] and nonunitary evolutions [19] would therefore not modify the conceptual framework but would still deserve further investigation [20].

Quantum Zeno dynamics and entanglement.—A second important consequence of the relation between the Fisher information and the Zeno dynamics is that τ_{QZ} Eq. (10) can be much smaller for entangled states than for separable states. To illustrate this point let's consider, as a simple example, the dynamics of N qubits governed by a local

Hamiltonian $H = \omega \sum_{l=1}^{N} \vec{\sigma}_l \vec{n}_l$, where $\vec{\sigma}_l$ are Pauli matrices and $\vec{n}_l$ unit vectors. If the state is classically correlated, namely, it can be written as a convex combination of N qubits separable states:

$$\rho_{\rm sep} = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \ldots \otimes \rho_k^{(N)}, \quad (12)$$

then the Fisher information is bounded by $F \leq N\omega^2$ [22]. Therefore, for separable states, the quantum Zeno time scale is:

$$\tau_{\rm QZ} \geq \frac{2}{\omega\sqrt{mN}}. \quad (13)$$

On the other hand, the highest possible value of the Fisher information, saturable with a maximally entangled state, is $F \leq N^2\omega^2$. Therefore, a *sufficient* condition for the presence of entanglement is $F > N\omega^2$ [22]. This class of entangled states has a quantum Zeno time scale

$$\frac{2}{\omega N\sqrt{m}} \leq \tau_{\rm QZ} < \frac{2}{\omega\sqrt{Nm}}, \quad (14)$$

which can be smaller up to a factor $\sqrt{N}$ than the Zeno time of separable states. The consequence of (13) and (14), is that the Zeno dynamics can require a much higher rate of measurements if the state is entangled rather than separable.

This prediction can be tested experimentally with Mach-Zehnder (MZ) interferometers. A MZ interferometer is mathematically described by a unitary evolution which rotates the initial state in the Bloch sphere by an angle corresponding to the phase shift θ applied between the two arms of the interferometer [16] $|\psi(\theta)\rangle = e^{-iJ_y\theta}|\psi_{\rm inp}\rangle$. As $|\psi_{\rm inp}\rangle$ we consider a state made of N particles and the generator of the phase shift θ is $J_y = \frac{1}{2}\sum_{l=1}^{N} \sigma_y^{(l)}$. Let's consider m Mach-Zehnder interferometers sequentially connected so that the output state at the ports $\{c_j, d_j\}$ of the jth interferometer becomes the input state at the ports $\{a_{j+1}, b_{j+1}\}$ of the next one as in Fig. 1(a). In each interferometer we apply a phase shift θ/m. The rotation of the initial state is simply given by $|\psi(\theta)\rangle = \prod_{i=1}^{m} e^{-iJ_y\theta/m}|\psi_{\rm inp}\rangle$ which is obviously equivalent to a single MZ interferometer with a phase shift θ. We choose as input of the first MZ interferometer a mode-separable state $|\psi_{\rm inp}\rangle = |\psi_a\rangle|\psi_b\rangle$ [23]. In order to study the quantum Zeno dynamics, we cut the connections at the output ports $c_1, c_2, \ldots, c_m$. We can leave undetected the particles which might exit those ports. The input ports $a_1, a_2, \ldots, a_m$ are instead injected with a state identical to the initial one $|\psi_a\rangle$, see Fig. 1(b). What is the final state of the interferometer? If the number m of interferometers is sufficiently large (the phase shift θ/m suffficiently small), than the the state in output is equal to the state entering the input ports of the the first MZ interferometer.

How large does m have to be? Let's consider first the case of a classically particle correlated input $|\psi_{\rm inp}\rangle = |0\rangle_a|N\rangle_b$ [23]. This state has a Fisher information $F = N$.

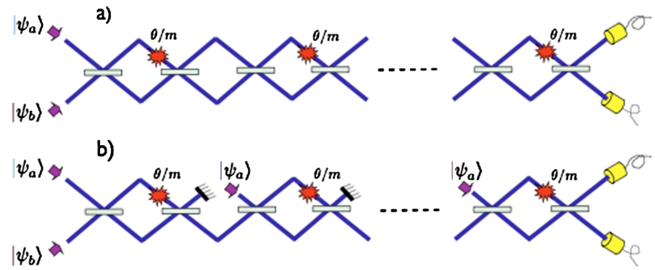

FIG. 1 (color online). Two different setups consisting of a sequence of m Mach-Zehnder interferometers. The phase shift in each MZ interferometer is θ/m. The input state is the product $|\psi_a\rangle|\psi_b\rangle$. (a) The MZ interferometers are sequentially connected so that the output of each interferometer becomes the input of the next one. (b) Only one output port becomes the input of the next interferometer. The second output port remains disconnected, while the corresponding input port of the next MZ interferometer is injected with $|\psi_a\rangle$.

The survival probability to have in output the same state as in input, i.e., to detect N and 0 particles at the output ports of the last interferometer, is $P(\theta) \simeq 1 - \frac{N}{4m}\theta^2$, cf. Eq. (9) [16]. When the number of measurements $m \gg N$ we can observe, in agreement with Eq. (13), the Zeno effect.

Entanglement changes the scenario. Let's inject the previously unused ports with a Fock state having the same number of particles as the second port, $|\psi_{\rm inp}\rangle = |N/2\rangle_a|N/2\rangle_b$. This is a particle entangled state with the Fisher information $F = N^2/2 + N$ [23]. The survival probability to detect $N/2$ and $N/2$ particles at the output ports of the last interferometer is $P(\theta) \simeq 1 - \frac{N^2}{8m}\theta^2$ [16]. In contrast to the previous case and in agreement with Eq. (14), we observe Zeno dynamics only when the number of measurements (and MZ interferometers) is $m \gg N^2$. In other words, with entangled states the phase shift θ/m in each interferometer might need to be smaller by a factor $1/\sqrt{N}$ with respect to classically correlated states. The Fisher information of the two states $|0\rangle_a|N\rangle_b$ and $|N/2\rangle_a|N/2\rangle_b$ with $N = 4$ has been measured experimentally in Ref. [24] along with the implementation of a full phase estimation analysis.

It is worth emphasizing that equivalent protocols can be implemented when the total number of particles is not fixed and/or with cold atoms and resonant pulses between internal hyperfine levels. To illustrate this point, let's briefly consider the experimental setup of Ref. [8], which was based on the proposal [25] considering a three-level atom, see Fig. (2). The two modes a, b of the Ramsey interferometer are two sublevels of the ground $2S_{1/2}$ state of a Be^+ ion. The third level c is a sublevel of the $2P_{3/2}$ excited state which can decay only in to the level a. In the experiment, the level b was populated with N ions and a resonant radio frequency was applied to drive the atoms to the level a. The nondestructive measurements of the number of particles populating the level a are carried out by short laser pulses

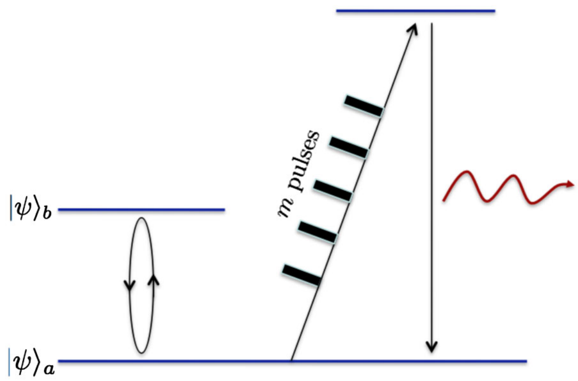

FIG. 2 (color online). Energy level diagram of the experiment [8]. A resonant radio-frequency magnetic field couples the a and b modes. A sequence of m laser pulses resonantly couples the levels a and c. The ions in the level c decay only in the level a with induced fluorescence proportional to the number of particles populating the mode a.

resonant with the $a \rightarrow c$ transition. The Zeno dynamics is observed when the number of such measurements is sufficiently large. If, on the other hand, the initial state is a twin-Fock or a spin-squeezed state, the number of measurements needed to observe the Zeno dynamics would be quite larger. Atomic spin-squeezed and twin matter waves have been recently created with dilute atomic Bose-Einstein condensates [26].

Quantum Zeno and Cramer-Rao lower bound.—Motivated by the previous interferometric examples, we digress and briefly discuss the connection between distinguishability and the theory of parameter estimation. Let's consider the interval τ as an unknown parameter which has to be estimated from the results of m measurements on the state $e^{-i\tau\sum\mathcal{H}_k}[\otimes_{k=0}^{m}\rho(k\tau)]e^{i\tau\sum\mathcal{H}_k}$. This evolved state is indistinguishable from the initial state if and only if the parameter τ is smaller than the estimation noise arising from the stochastic set of measurement results $\{\text{yes}, \text{no}\}$. A fundamental lower bound for the noise is provided by the Cramer-Rao relation $\Delta\tau_{\text{CR}} = 1/\sqrt{mF}$ which provides a natural condition for indistinguishability (and, therefore, for Zeno dynamics) in terms of a signal-to-noise ratio $\tau/\Delta\tau_{\text{CR}} \ll 1$ [27].

Some final remarks for the case of pure states.—Generally speaking, the value of the Fisher information depends on our choice of the observable. The highest value of the Fisher information, obtained with an optimal choice of the measurement apparatus (the one that can better discriminate neighboring states), is referred to in the literature as the quantum Fisher [28]. For the one-dimensional case Eq. (1), the quantum Fisher is given by $|\langle\psi(t)|\psi(t+\delta t)\rangle|^2 = 1 - F_q(\delta t)^2$ and is precisely $F_q = 4\Delta^2\mathcal{H}$ [29]. Therefore, the survival probability can be written as: $P(t) \sim e^{-(\tau/2\Delta t_{\text{QCR}})^2}$ where $\Delta t_{\text{QCR}} = 1/\sqrt{mF_q}$ is the quantum Cramer-Rao bound. Notice also that the Cramer-Rao can be written as an uncertainty relation that, for pure states, is $\Delta t_{\text{QCR}}\Delta\mathcal{H} \geq 1/2\sqrt{m}$. This relation has of course a different physical meaning than a Heisenberg uncertainty relation since Δt_{QCR} is the mean square fluctuation of estimated parameters rather than of measurement results of an Hermitian operators [30]. In particular, such a parameter can be estimated with arbitrary precision by just increasing m. The fact that the quantum Zeno effect is related to a parameter-based uncertainty relation also clarifies the important point (which occasionally raises some controversy) that there are no intrinsic limits, in principle, on the rate of measurements: the value θ/m in the previous interferometry examples can be arbitrarily small.

Conclusions.—In the classical world, indistinguishability is the consequence of ignorance, which can be solved by collecting results of measurements. This is not always the case in the quantum word. In the Zeno dynamics, the projective measurements freeze indistinguishability.

There are different technologies which are based on efficiently distinguished quantum states. Our results can therefore be extended and applied in various contexts such as, for instance, in quantum control theories, when searching the optimal path to generate a target quantum state [31], in the conditions for adiabatic approximations [32] and applications in adiabatic quantum computation [33], and in the estimation of the speed limits of quantum computation [34].

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[1] B. Misra and E. C. G. Sudharshan, J. Math. Phys. Sci. **18**, 756 (1977).
[2] The Zeno paradoxes were reported in Aristotle's Physics.
[3] For a review see: P. Facchi and S. Pascazio, J. Phys. A **41**, 493001 (2008).
[4] K. Koshino and A. Shimizu, Phys. Rep. **412**, 191 (2005).
[5] L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998).
[6] J. D. Franson, B. C. Jacobs, and T. B. Pittman, Phys. Rev. A **70**, 062302 (2004).
[7] O. Hosten, M. T. Rakher, J. T. Barreiro, N. A. Peters, and P. G. Kwiat, Nature (London) **439**, 949 (2006).
[8] W. M. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, Phys. Rev. A **41**, 2295 (1990).
[9] P. Kwiat, H. Weinfurter, T. Herzog, A. Zeilinger, and M. A. Kasevich, Phys. Rev. Lett. **74**, 4763 (1995).
[10] M. C. Fischer, B. Gutierrez-Medina, and M. G. Raizen Phys. Rev. Lett. **87**, 040402 (2001).
[11] E. W. Streed, J. Mun, M. Boyd, G. K. Campbell, P. Medley, W. Ketterle, and D. E. Pritchard, Phys. Rev. Lett. **97**, 260402 (2006).

[12] A. Peres, J. Phys. A **48**, 931 (1980); D. Home and M. A. B. Whitaker, J. Phys. A **25**, 657 (1992).
[13] A simple physical argument to understand why the survival probability shrinks when increasing $\Delta^2\mathcal{H}$ is that the Hamiltonian is the generator of the time evolution: the larger is its variance, the quicker the time evolved state $|\psi(\tau)\rangle$ and the initial $|\psi_0\rangle$ come to be different.
[14] P. Facchi and S. Pascazio, Phys. Rev. Lett. **89**, 080401 (2002).
[15] W. K. Wootters, Phys. Rev. D **23**, 357 (1981).
[16] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.109.150410 for the calculation of the survival probability, the Fisher information and the Zeno effect with separable and entangled states.
[17] Needless to say, the one-dimensional QZ Eq. (1) is recovered with $\rho_0 = |\psi_0\rangle\langle\psi_0|$ and projectors $\Pi = |\psi_0\rangle\langle\psi_0|$.
[18] These probabilities can be retrieved experimentally; L. Pezzé, A. Smerzi, G. Khoury, J. F. Hodelin, and D. Bouwmeester, Phys. Rev. Lett. **99**, 223602 (2007).
[19] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, Nature Phys. **7**, 406 (2011).
[20] We also notice that our analysis focuses on closed systems. Open systems can have a richer dynamics (for instance, the survival probability might have non-integer powers in the expansion) [4,21].
[21] A. G. Kofman and G. Kurizki, Phys. Rev. Lett. **87**, 270405 (2001); Erez, G. Gordon, M. Nest, and G. Kurizki, Nature (London) **452**, 724 (2008).
[22] L. Pezzé and A. Smerzi, Phys. Rev. Lett. **102**, 100401 (2009).
[23] We notice that a mode-separable state can, either, be classically correlated or entangled in the qubits according to the definition Eq. (12). For instance, with $N = 2$, the mode-separable twin-Fock state $|1_a\rangle|1_b\rangle$ is particle entangled since in the first quantization it is $(|\uparrow_1\rangle|\downarrow_2\rangle + |\downarrow_1\rangle|\uparrow_2\rangle)/\sqrt{2}$, where 1 and 2 label the particles and $|\uparrow\rangle, |\downarrow\rangle$ are the eigenstates of σ_z. The state $|0_a\rangle|2_b\rangle$ is particle separable since in first quantization is $|\uparrow_1\rangle|\uparrow_2\rangle$.
[24] R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, P. Hyllus, A. Smerzi, and L. Pezzé, Phys. Rev. Lett. **107**, 080504 (2011).
[25] R. J. Cook, Phys. Scr. **T21**, 49 (1988).
[26] C. Gross, T. Zibold, E. Nicklas, J. Estève, and M. K. Oberthaler, Nature (London) **464**, 1165 (2010); M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. Treutlein, Nature (London) **464**, 1170 (2010); B. Lücke *et al.*, Science **334**, 773 (2011); R. Bücker, J. Grond, S. Manz, T. Berrada, T. Betz, C. Koller, U. Hohenester, T. Schumm, A. Perrin, and J. Schmiedmayer, Nature Phys. **7**, 608 (2011).
[27] Let us look at QZ by following the dynamics at *fixed* intervals τ and increasing the number of measurements (which corresponds to consider increasing final times). Let's consider a value of τ much smaller than the noise in order to observe Zeno at $t = \tau m$ after a given $m \gg 1$ number of measurements. At larger times, since the noise decreases as $\Delta\tau_{\rm CR} \sim 1/\sqrt{m}$, inevitably we reach the point where $\tau \sim \Delta\tau_{cr}$ at a critical number of measurements $m_{\rm crit} \sim 1/\tau^2 F$. This defines the critical time $t_{\rm crit} \sim \sqrt{m_{\rm crit}/F} = 1/\tau F$ where the evolved state can be recognized to be different from the initial state and the dynamics ceases to be Zeno [the expansion bringing to Eq. (9) breaks down]. At even larger times, when $\tau > \Delta\tau_{\rm CR}$, the anti-Zeno effect might accelerate the dynamics [35].
[28] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[29] The survival probability after a single measurement is given by the fidelity which can be related to the Fubini-Study metric; A. K. Pati, Phys. Lett. A **215**, 7 (1996).
[30] S. L. Braunstein, C. M. Caves, and G. J. Milburn, Ann. Phys. (Leipzig) **247**, 135 (1996).
[31] T. Caneva, M. Murphy, T. Calarco, R. Fazio, S. Montangero, V. Giovannetti, and G. E. Santoro, Phys. Rev. Lett. **103**, 240501 (2009).
[32] S. Boixo and R. D. Somma, Phys. Rev. A **81**, 032308 (2010).
[33] E. Farhi *et al.*, arXiv:quant-ph/0001106.
[34] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. A**67**, 052109 (2003); L. B. Levitin and T. Toffoli, Phys. Rev. Lett. **103**, 160502 (2009).
[35] A. G. Kofman and G. Kurizki, Nature (London) **405**, 546 (2000); P. Facchi, H. Nakazato, and S. Pascazio, Phys. Rev. Lett. **86**, 2699 (2001).