Complete Tenth-Order QED Contribution to the Muon $g - 2$

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We report the result of our calculation of the complete tenth-order QED terms of the muon $g - 2$. Our result is $a_\mu^{(10)} = 753.29\ (1.04)$ in units of $(\alpha/\pi)^5$, which is about 4.5 s.d. larger than the leading-logarithmic estimate 663(20). We also improve the precision of the eighth-order QED term of a_μ, obtaining $a_\mu^{(8)} = 130.8794\ (63)$ in units of $(\alpha/\pi)^4$. The new QED contribution is $a_\mu(\text{QED}) = 116\,584\,718\,951\,(80) \times 10^{-14}$, which does not resolve the existing discrepancy between the standard-model prediction and measurement of a_μ.

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The anomalous magnetic moment a_μ of the muon has been studied extensively both experimentally and theoretically since it provides one of the promising paths in exploring possible new physics beyond the standard model. For this purpose it is crucial to know the prediction of the standard model as precisely as possible.

On the experimental side the current world average of the measured a_μ is [1,2]:

$$a_\mu(\text{exp}) = 116\,592\,089\,(63) \times 10^{-11}\ [0.5\ \text{ppm}]. \quad (1)$$

New experiments designed to improve the precision further are being prepared at Fermilab [3] and J-PARC [4].

In the standard model, a_μ can be divided into electromagnetic, hadronic, and electroweak contributions

$$a_\mu = a_\mu(\text{QED}) + a_\mu(\text{hadronic}) + a_\mu(\text{electroweak}). \quad (2)$$

At present a_μ (hadronic) is the largest source of theoretical uncertainty. The uncertainty comes mostly from the $O(\alpha^2)$ hadronic vacuum-polarization (v.p.) term, α being the fine-structure constant. The lattice QCD simulations have attempted to evaluate this contribution [5–10]. At present, most accurate evaluations must rely on the experimental information. Three types of measurements are available for this purpose: (1) $e^+e^- \to \text{hadrons}$, (2) $\tau^\pm \to \nu + \pi^\pm + \pi^0$, (3) $e^+e^- \to \gamma + \text{hadrons}$. These processes have been investigated intensely by many groups [11–13]. We list here one of them [13]:

$$a_\mu(\text{had.v.p.}) = 6949.1(37.2)_{\text{exp}}(21.0)_{\text{rad}} \times 10^{-11}, \quad (3)$$

which overlaps other values based on the e^+e^- data [11,12] and makes the standard-model prediction closest to the experiment (1). The next-to-leading-order (NLO) hadronic vacuum-polarization contribution is also known [13]:

$$a_\mu(\text{NLO had.v.p.}) = -98.4(0.6)_{\text{exp}}(0.4)_{\text{rad}} \times 10^{-11}. \quad (4)$$

The hadronic light-by-light scattering contribution (l-l) is of similar size as a_μ (NLO had.v.p.), but has a much larger theoretical uncertainty [14–17]

$$a_\mu(\text{had.}\,l\text{-}l) = 116(40) \times 10^{-11}, \quad (5)$$

where the uncertainty 40×10^{-11} covers almost all values obtained in different publications.

The electroweak contribution has been calculated up to 2-loop order [18–21]:

$$a_\mu(\text{weak}) = 154(2) \times 10^{-11}. \quad (6)$$

Since this uncertainty is 30 times smaller than the experimental precision of (1), it can be regarded as known precisely.

The primary purpose of this letter is to report the complete numerical evaluation of *all* tenth-order QED contribution to a_μ. It leads to a sizable reduction of the uncertainty of the previous estimate by the leading-log approximations [22,23]. We have also improved the numerical precision of the eighth-order QED contribution including the newly evaluated tau-lepton contribution. Together they represent a significant reduction in the theoretical uncertainty of the QED part of a_μ.

The QED contribution to a_μ can be evaluated by the perturbative expansion in α/π:

$$a_\mu(\text{QED}) = \sum_{n=1}^{\infty}\left(\frac{\alpha}{\pi}\right)^n a_\mu^{(2n)}, \quad (7)$$

where $a_\mu^{(2n)}$ is finite thanks to the renormalizability of QED and can be written as

$$a_\mu^{(2n)} = A_1^{(2n)} + A_2^{(2n)}(m_\mu/m_e) + A_2^{(2n)}(m_\mu/m_\tau) + A_3^{(2n)}(m_\mu/m_e, m_\mu/m_\tau). \quad (8)$$

$A_1^{(2n)}$ is independent of mass and universal for all leptons. $A_1^{(2)}$, $A_1^{(4)}$ and $A_1^{(6)}$ are known exactly [24–27]. Mass dependence is known analytically for $A_2^{(2n)}$ and $A_3^{(2n)}$ for $n = 2, 3$ [28–32]. We reevaluated them using the latest values of the muon-electron mass ratio $m_\mu/m_e = 206.768\,284\,3\ (52)$ and/or the muon-tau mass ratio $m_\mu/m_\tau = 5.946\,49\,(54) \times 10^{-2}$ [33]. In the same order of terms as shown on the right-hand side of (8), the results are summarized as follows:

$$\begin{aligned} a_\mu^{(2)} &= 0.5, \\ a_\mu^{(4)} &= -0.328\,478\,965\,579\ldots + 1.094\,258\,312\,0\,(83) \\ &\quad + 0.780\,79(15) \times 10^{-4} \\ &= 0.765\,857\,425\,(17), \\ a_\mu^{(6)} &= 1.181\,241\,456\ldots + 22.868\,380\,04(23) \\ &\quad + 0.360\,70(13) \times 10^{-3} + 0.527\,76(11) \times 10^{-3} \\ &= 24.050\,509\,96\,(32). \end{aligned} \tag{9}$$

The value of $a_\mu^{(8)}$ has been obtained mostly by numerical integration [34–36]. They arise from 13 gauge-invariant sets whose representative diagrams are shown in Fig. 1. We have reevaluated some of them for further check and improvement of numerical precision. The results for the mass-dependent terms are summarized in Table I.

From the data listed in Table I and the value of $A_1^{(8)}$ from Refs. [35–37], we obtain the following value for the eighth-order QED contribution $a_\mu^{(8)}$:

$$\begin{aligned} a_\mu^{(8)} &= -1.9106\,(20) + 132.685\,2\,(60) \\ &\quad + 0.042\,34\,(12) + 0.062\,72\,(4) \\ &= 130.879\,6(63). \end{aligned} \tag{10}$$

Over the period of more than nine years we have numerically evaluated all 32 gauge-invariant sets of diagrams that contribute to $a_\mu^{(10)}$ [22,37–40], whose representative diagrams are shown in Fig. 2. The results for mass-dependent terms are summarized in Table II. Some simple diagrams were evaluated analytically or in the asymptotic expansion in m_μ/m_e [41–45]. The results are consistent with our numerical ones.

From the data listed in this Table and the value of $A_1^{(10)}$ from Ref. [37], we obtain the complete tenth-order result:

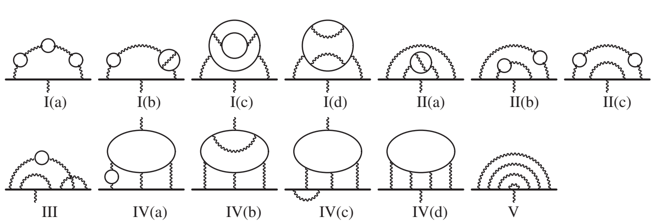

FIG. 1. Vertex diagrams representing 13 gauge-invariant subsets contributing to the lepton $g-2$ at the eighth-order. Solid and wavy lines represent lepton and photon lines, respectively.

$$\begin{aligned} a_\mu^{(10)} &= 9.168\,(571) + 742.18\,(87) - 0.068\,(5) + 2.011\,(10) \\ &= 753.29\,(1.04). \end{aligned} \tag{11}$$

The uncertainty 1.04 is attributed entirely to the statistical fluctuation in the Monte Carlo integration of Feynman amplitudes by VEGAS [46]. This is 20 times more precise than the previous estimate, 663 (20), obtained in the leading-logarithmic approximation [22]. Note also that (11) is about 4.5 s.d. larger than the leading-log estimate. This is mainly because we had underestimated the magnitude of the contribution of the Set III(a). The numerical values of $(\alpha/\pi)^{(n)} a_\mu^{(2n)}$ for $n = 1, 2, \cdots, 5$ are summarized in Table III.

In order to evaluate a_μ (QED) using (7), a precise value of α is needed. At present, the best non-QED α is the one obtained from the measurement of h/m_{Rb} [47], combined with the very precisely known Rydberg constant and m_{Rb}/m_e [33]:

$$\alpha^{-1}(\mathrm{Rb}) = 137.035\,999\,049\,(90)\,[0.66\ \mathrm{ppb}]. \tag{12}$$

Actually, we have a more precise value of α which is derived from the measurement [48,49] and theory of the electron $g-2$ [37]:

$$\alpha^{-1}(a_e) = 137.035\,999\,172\,7\,(68)(46)(19)(331)\,[0.25\ \mathrm{ppb}], \tag{13}$$

where the first three uncertainties are due to the eighth-order term, tenth-order term, and the hadronic and electroweak terms, involved in the evaluation of a_e. The fourth uncertainty comes from the measurement of a_e. At present the difference between (12) and (13) is much smaller than the current uncertainty in the measurement of a_μ so that one may use either one of these two. However, some caution must be exercised to employ $\alpha^{-1}(a_e)$ to calculate a_μ, when more accurate experiment of a_μ becomes

TABLE I. The eighth-order mass-dependent QED contribution from 12 gauge-invariant groups to muon $g-2$, whose representatives are shown in Fig. 1. The mass-dependence of $A_3^{(8)}$ is $A_3^{(8)}(m_\mu/m_e, m_\mu/m_\tau)$. The value with an asterisk is in agreement with the result in Ref. [51].

group	$A_2^{(8)}(m_\mu/m_e)$	$A_2^{(8)}(m_\mu/m_\tau)$	$A_3^{(8)}$
I(a)	7.745 47 (42)	0.000 032 (0)	0.003 209 (0)
I(b)	7.582 01 (71)	0.000 252 (0)	0.002 611 (0)
I(c)	1.624 307 (40)	0.000 737 (0)	0.001 807 (0)
I(d)	−0.229 82 (37)	0.000 368 (0)	0
II(a)	−2.778 88 (38)	−0.007 329 (1)	0
II(b)	−4.552 77 (30)	−0.002 036 (0)	−0.009 008 (1)
II(c)	−9.341 80 (83)	−0.005 246 (1)	−0.019 642 (2)
III	10.7934 (27)	0.045 04 (14)	0
IV(a)	123.785 51 (44)	0.038 513 (11)	0.083 739 (36)
IV(b)	−0.4170 (37)	0.006 106(31)*	0
IV(c)	2.9072 (44)	−0.018 23 (11)	0
IV(d)	−4.432 43 (58)	−0.015 868 (37)	0

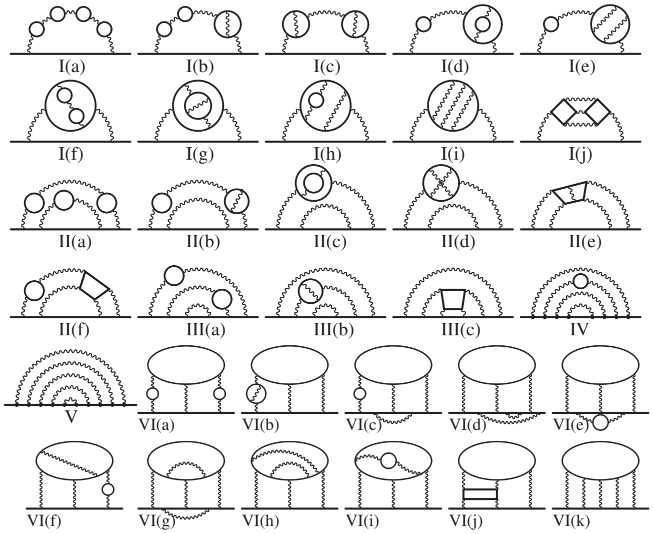

FIG. 2. Self-energy-like diagrams representing 32 gauge-invariant subsets contributing to the lepton $g-2$ at the tenth order. Solid lines represent lepton lines propagating in a weak magnetic field.

available, because theoretical calculation of a_e is strongly correlated with that of a_μ.

Substituting (9)–(11) in Eq. (7) and using (12), we obtain

$$a_\mu(\text{QED, Rb}) = 116\,584\,718\,951\;(9)(19)(7)(77) \times 10^{-14}, \tag{14}$$

where the uncertainties are from the lepton mass ratios, the eighth-order term, the tenth-order term, and the value of α in (12), respectively. If we use the value of α in (13) instead, we get

$$a_\mu(\text{QED}, a_e) = 116\,584\,718\,846\;(9)(19)(7)(30) \times 10^{-14}. \tag{15}$$

Note that the uncertainties of the lepton mass ratios, the eighth-order term, the tenth-order terms, and $\alpha(a_e)$ are improved by factors 1.7, 1.3, 20, and 1.5, respectively, compared with $a_\mu(\text{QED}, a_e)$ given in Eq. (99) of Ref. [50].

The difference between (14) and (15) is less than 1.2×10^{-12} so that we may use either one as far as comparison with the current experimental data is concerned.

In view of the rather large value of $A_2^{(10)}(m_\mu/m_e)$ one might wonder how large $A_2^{(12)}(m_\mu/m_e)$ might be. As a matter of fact it is not difficult to estimate its size. For this purpose note that the dominant contribution to $A_2^{(8)}(m_\mu/m_e)$ comes from the Group IV(a) and the dominant contribution to $A_2^{(10)}(m_\mu/m_e)$ comes from the Set VI (a). Both are integrals obtained by inserting several second-order vacuum-polarization loops Π_2 into the virtual photon lines of the sixth-order diagram $A_2^{(6)}(m_\mu/m_e; l\text{-}l)$ which contains a light-by-light scattering electron loop. Analogously the leading contribution to the twelfth-order term will come from insertion of three Π_2's in $A_2^{(6)}(m_\mu/m_e; l\text{-}l)$, namely,

$$A_2^{(12)}(m_\mu/m_e) \sim A_2^{(6)}(m_\mu/m_e; l\text{-}l) \times \left\{\frac{2}{3}\ln\left(\frac{m_\mu}{m_e}\right) - \frac{5}{9}\right\}^3 \times 10 \tag{16}$$

$$A_2^{(12)}(m_\mu/m_e)\left(\frac{\alpha}{\pi}\right)^6 \sim 0.8 \times 10^{-12}, \tag{17}$$

noting that $A_2^{(6)}(m_\mu/m_e; l\text{-}l) \sim 20$ and the factor 10 accounts for the possible ways of insertion of Π_2. Including the contribution of other diagrams, the size of the 12th-order term might be as large as 10^{-12}. This is larger than the uncertainty of the 10th-order term in (14) so that it would be desirable to obtain at least a crude evaluation of this term.

Adding (3)–(6) and (14), and using α from (12), the theoretical value of a_μ in the standard model is given by

$$a_\mu(\text{SM}) = 116\,591\,840\,(59) \times 10^{-11}. \tag{18}$$

TABLE II. Tenth-order mass-dependent contribution to the muon $g-2$ from 31 gauge-invariant subsets shown in Fig. 2. The mass-dependence of $A_3^{(10)}$ is $A_3^{(10)}(m_\mu/m_e, m_\mu/m_\tau)$.

set	$A_2^{(10)}(m_\mu/m_e)$	$A_2^{(10)}(m_\mu/m_\tau)$	$A_3^{(10)}$
I(a)	22.566 973 (3)	0.000 038 (0)	0.017 312 (1)
I(b)	30.667 091 (3)	0.000 269 (0)	0.020 179 (1)
I(c)	5.141 395 (1)	0.000 397 (0)	0.002 330 (0)
I(d)	8.8921 (11)	0.000 388 (0)	0.024 487 (2)
I(e)	−0.9312 (24)	0.000 232 (0)	0.002 370 (0)
I(f)	3.685 049 (90)	0.002 162 (0)	0.023 390 (2)
I(g)	2.607 87 (72)	0.001 698 (0)	0.002 729 (1)
I(h)	−0.5686 (11)	0.000 163 (1)	0.001 976 (3)
I(i)	0.0871 (59)	0.000 024 (0)	0
I(j)	−1.263 72 (14)	0.000 168 (1)	0.000 110 (5)
II(a)	−70.4717 (38)	−0.018 882 (8)	−0.290 853 (85)
II(b)	−34.7715 (26)	−0.035 615 (20)	−0.127 369 (60)
II(c)	−5.385 75 (99)	−0.016 348 (14)	−0.040 800 (51)
II(d)	0.4972 (65)	−0.007 673 (14)	0
II(e)	3.265 (12)	−0.038 06 (13)	0
II(f)	−77.465 (12)	−0.267 23 (73)	−0.502 95 (68)
III(a)	109.116 (33)	0.283 000 (32)	0.891 40 (44)
III(b)	11.9367 (45)	0.143 600 (10)	0
III(c)	7.37 (15)	0.1999 (28)	0
IV	−38.79 (17)	−0.4357 (25)	0
VI(a)	629.141 (12)	0.246 10 (18)	2.3590 (18)
VI(b)	181.1285 (51)	0.096 522 (93)	0.194 76 (26)
VI(c)	−36.58 (12)	−0.2601 (28)	−0.5018 (89)
VI(d)	−7.92 (60)	0.0818 (17)	0
VI(e)	−4.32 (14)	−0.035 94 (32)	−0.1122 (24)
VI(f)	−38.16 (15)	0.043 47 (85)	0.0659 (31)
VI(g)	6.96 (48)	−0.044 51 (96)	0
VI(h)	−8.55 (23)	0.004 85 (46)	0
VI(i)	−27.34 (12)	−0.003 45 (33)	−0.0027 (11)
VI(j)	−25.505 (20)	−0.011 49 (33)	−0.016 03 (58)
VI(k)	97.123 (62)	0.002 17 (16)	0

TABLE III. Contributions to muon $g-2$ from QED perturbation term $a_\mu^{(2n)}(\alpha/\pi)^n \times 10^{11}$. They are evaluated with two values of the fine-structure constant determined by the Rb experiment and by the electron $g-2$ (a_e).

order	with α^{-1}(Rb)	with $\alpha^{-1}(a_e)$
2	116 140 973.318 (77)	116 140 973.213 (30)
4	413 217.6291 (90)	413 217.6284 (89)
6	30 141.902 48 (41)	30 141.902 39 (40)
8	381.008 (19)	381.008 (19)
10	5.0938 (70)	5.0938 (70)
a_μ(QED) $\times 10^{11}$	116 584 718.951 (80)	116 584 718.846 (37)

We have therefore

$$a_\mu(\text{exp}) - a_\mu(\text{SM}) = 249(87) \times 10^{-11}. \quad (19)$$

The size of discrepancy between theory and experiment has not changed much, since the tenth-order QED contribution is not a significant source of theoretical uncertainties. Let us emphasize, however, that the complete calculation of $a_\mu^{(10)}$ enables us to concentrate on improving the precision of the hadronic contributions.

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