Invariant Beta Ensembles and the Gauss-Wigner Crossover

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We define a new diffusive matrix model converging toward the β-Dyson Brownian motion for all $\beta \in [0, 2]$ that provides an explicit construction of beta ensembles of random matrices that is invariant under the orthogonal or unitary group. For small values of β, our process allows one to interpolate smoothly between the Gaussian distribution and the Wigner semicircle. The interpolating limit distributions form a one parameter family that can be explicitly computed. This also allows us to compute the finite-size corrections to the semicircle.

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Since Wigner's initial intuition that the statistical properties of the eigenvalues of random matrices should provide a good description of the excited states of complex nuclei, random matrix theory has become one of the prominent fields of research, at the boundary between atomic physics, solid state physics, statistical mechanics, statistics, probability theory, and number theory [1–3]. It is well known that the joint distribution of the eigenvalues of a large Gaussian random matrix can be expressed as the Boltzmann-Gibbs equilibrium weight of a one-dimensional repulsive Coulomb gas confined in an harmonic well. However, the effective ''inverse temperature'' β of the system cannot take arbitrary values but is quantized (in units of the repulsive Coulomb potential). Depending on the symmetry of the random matrix, only three values are allowed $\beta = 1$ for symmetric real matrices, $\beta = 2$ for Hermitian matrices, and $\beta = 4$ for the symplectic ensemble. This is known as Dyson's ''threefold way''. The existence of matrix ensembles that would lead to other, possibly continuous, values of β, is a very natural question, and the quest for such ensembles probably goes back to Dyson himself. Ten years ago, Dumitriu and Edelman [4] have proposed an explicit construction of tridiagonal matrices with nonidentically distributed elements whose joint law of the eigenvalues is the one of beta ensembles for general β. Another construction is proposed in [1] (see also [5]) and uses a bordering procedure to construct recursively a sequence of matrices with eigenvalues distributed as beta ensembles. This construction gives not just the eigenvalue probability density of one matrix of the sequence but also the joint eigenvalue probability density of all matrices. This has lead to a renewed interest for those ensembles that have connections with many problems, both in physics and in mathematics; see, e.g., [3,6]. The aim of the Letter is to provide another construction of beta ensembles that is, at least to our eyes, natural and transparent and respects by construction the orthogonal or unitary symmetry [7]. Another motivation for our work comes from the recent development of free probability theory. ''Freeness'' for random matrices is the natural extension of independence for classical random variables. Very intuitively, two real symmetric matrices $\mathbf{A}, \mathbf{B}$ are mutually free in the large N limit if the eigenbasis of $\mathbf{B}$ can be thought of as a random rotation of the eigenbasis of $\mathbf{A}$ (see, e.g., [8] for an accessible introduction to freeness and for more rigorous statements). ''Free convolution'' then allows one to compute the eigenvalue distribution of the sum $\mathbf{A} + \mathbf{B}$ from the eigenvalue distribution of $\mathbf{A}$ and $\mathbf{B}$, much in the same way as convolution allows one to compute the distribution of the sum of two independent random variables. In this context, the Wigner semicircle distribution appears as the limiting distribution for the sum of a large number of free random matrices, exactly as the Gaussian is the limiting distribution for the sum of a large number of IID (independent and identically distributed) random variables. A natural question, from this perspective, is whether one can build a natural framework that interpolates between these two limits.

Let us first recall Dyson's Brownian motion construction of the Gaussian orthogonal ensemble (GOE) [9] (for the sake of simplicity, we will only consider here extensions of the $\beta = 1$ ensemble, but similar considerations hold for $\beta = 2$ Hermitian matrices, see [10] for full details). It is defined as the real $N \times N$ symmetric matrix process $\mathbf{M}(t)$ solution of the stochastic differential equation (SDE)

$$d\mathbf{M}(t) = -\tfrac{1}{2}\mathbf{M}(t)dt + d\mathbf{H}(t), \qquad (1)$$

where $d\mathbf{H}(t)$ is a symmetric Brownian increment [i.e., a symmetric matrix whose entries above the diagonal are independent Brownian increments with variance $\langle d\mathbf{H}_{ij}^2(t)\rangle = \frac{1}{2}(1 + \delta_{ij})dt$]. Standard second order perturbation theory allows one to write the evolution equation for the eigenvalues λ_i of the matrix $\mathbf{M}(t)$,

$$d\lambda_i = -\frac{1}{2}\lambda_i dt + \frac{1}{2}\sum_{j\neq i}\frac{dt}{\lambda_i - \lambda_j} + db_i, \qquad (2)$$

where $b_i(t)$ are independent standard Brownian motions. This defines Dyson's Coulomb gas model, i.e., "charged" particles on a line, with positions λ_i, interacting via a logarithmic potential, subject to some thermal noise and confined by a harmonic potential. One can deduce from the above equation the Fokker-Planck equation for the joint density $P(\{\lambda_i\}, t)$, for which the stationary joint probability density function (PDF) is readily found to be

$$P^*(\{\lambda_i\}) = Z\prod_{i<j}|\lambda_i - \lambda_j|^\beta \exp\left[-\frac{1}{2}\sum_i \lambda_i^2\right], \qquad (3)$$

with $\beta \equiv 1$ and where Z is a normalization factor. The above expression is the well-known joint distribution of the eigenvalues of an $N \times N$ random GOE matrix. The Wigner distribution can be recovered either by a careful analysis of the mean marginal univariate distribution $\rho(\lambda) = \int \cdots \int d\lambda_2 \cdots d\lambda_N P^*(\lambda = \lambda_1, \lambda_2, \ldots, \lambda_N)$ in the large N limit [11] or by using the above SDE (2) to derive a dynamical equation for the Stieltjes transform $G(z, t)$ of $\rho(\lambda, t)$,

$$G(z, t) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\lambda_i(t) - z}, \qquad z \in \mathbb{C}. \qquad (4)$$

With this scaling, the spectrum is spread out in a region of width of order $\sqrt{N}$, and therefore, $z \sim \sqrt{N}$ and $G \sim 1/\sqrt{N}$. Applying Itô's formula to $G(z, t)$ and using Eq. (2), we obtain the following Burgers equation for G [12]:

$$2\frac{\partial\langle G\rangle}{\partial t} = \frac{\alpha N}{2}\frac{\partial\langle G\rangle^2}{\partial z} + \frac{\partial z\langle G\rangle}{\partial z} + (2-\alpha)\frac{1}{2}\frac{\partial^2\langle G\rangle}{\partial z^2}, \qquad (5)$$

where α is introduced for later convenience, with $\alpha = 1$ for now. Note that we have neglected in Eq. (5) a term of order $N^{-5/2}$. Indeed, in agreement with [2], $\langle G^2\rangle - \langle G\rangle^2 \sim N^{-3}$. The neglected term is thus $1/N$ smaller than the diffusion term in Eq. (5).

For large N, the last (diffusion) term of Eq. (5) is of order $1/N$ smaller than the other ones. To leading order, the stationary solution (where the time derivative is set to 0) can be integrated with respect to z,

$$\tfrac{1}{2}\alpha N G_\infty^2(z) + zG_\infty(z) = -1, \qquad (6)$$

where the integration constant comes from the boundary condition $G(z) \sim -1/z$ when $z \to \infty$. It is then easy to solve this equation to find the Stieltjes transform that indeed corresponds to the Wigner semicircle density:

$$G_\infty(z) = \frac{1}{\alpha N}\left[\sqrt{z^2 - 2\alpha N} - z\right]$$
$$\to \rho(\lambda) = \frac{1}{\pi\alpha N}\sqrt{2\alpha N - \lambda^2}\,\mathbb{1}_{\{|\lambda|\le\sqrt{2\alpha N}\}}. \qquad (7)$$

Now let us turn to the central idea of the present Letter. In Dyson's construction, the extra Gaussian slice $d\mathbf{M}(t)$ that is added to $\mathbf{H}(t)$ is chosen to be independent of $\mathbf{M}(t)$ itself. The eigenbasis of $d\mathbf{H}(t)$ is a random rotation, taken uniformly over the orthogonal group. As mentioned above, this corresponds to free addition of matrices, and Eq. (5) can indeed be derived (for $N = \infty$) using free convolution [8]. If instead we choose to add a random matrix $d\mathbf{Y}(t)$ that is always diagonal in the same basis as that of $\mathbf{M}(t)$, the process becomes trivial. The diagonal elements of $\mathbf{M}(t)$ are all sums of IID random variables, and the eigenvalue distribution converges toward the Gaussian. The construction we propose is to alternate randomly the addition of a "free" slice and of a "commuting" slice. More precisely, our model is defined as follows: we divide time into small intervals of length $1/n$ and for each interval $[k/n;\ (k+1)/n]$, we choose independently Bernoulli random variables ϵ_k^n, $k \in \mathbb{N}$ such that $\mathbb{P}[\epsilon_k^n = 1] = p = 1 - \mathbb{P}[\epsilon_k^n = 0]$. Then, setting $\epsilon_t^n = \epsilon_{[nt]}^n$, our diffusive matrix process simply evolves as

$$d\mathbf{M}_n(t) = -\tfrac{1}{2}\mathbf{M}_n(t)dt + \epsilon_t^n d\mathbf{H}(t) + (1-\epsilon_t^n)d\mathbf{Y}(t), \qquad (8)$$

where $d\mathbf{H}(t)$ is a symmetric Brownian increment as above and where $d\mathbf{Y}(t)$ is a symmetric matrix that is codiagonalizable with $\mathbf{M}_n(t)$ (i.e., the two matrices have the same eigenvectors) but with a spectrum given by N independent Brownian increments of variance dt. It is clear that the eigenvalues of the matrix $\mathbf{M}_n(t)$ will cross at some points but only in intervals $[k/n; (k+1)/n]$ for which $\epsilon_k^n = 0$ (in the other intervals where they follow Dyson Brownian motion with parameter $\beta = 1$, it is well known that the repulsion is too strong and that collisions are avoided). In such a case, the eigenvalues are renumbered at time $t = (k+1)/n$ in increasing order.

Now, using again standard perturbation theory, it is easy to derive the evolution of the eigenvalues of $\mathbf{M}_n(t)$ denoted as $\lambda_1^n(t) \le \ldots \le \lambda_N^n(t)$,

$$d\lambda_i^n = -\frac{1}{2}\lambda_i^n dt + \frac{\epsilon_t^n}{2}\sum_{j\ne i}\frac{dt}{\lambda_i^n - \lambda_j^n} + db_i, \qquad (9)$$

where the b_i are independent Brownian motions also independent of the ϵ_k^n, $k \in \mathbb{N}$.

A mathematically rigorous derivation provided in [10] allows one to show that the scaling limits $\lambda_i(t)$, when $n \to \infty$, of the eigenvalues $\lambda_i^n(t)$ obey the following modified Dyson SDE:

$$d\lambda_i = -\frac{1}{2}\lambda_i dt + \frac{p}{2}\sum_{j\ne i}\frac{dt}{\lambda_i - \lambda_j} + db_i, \qquad (10)$$

with the additional ordering constraint $\lambda_1(t) \le \ldots \le \lambda_N(t)$ for all t. One of the difficulties of the proof comes from the fact that when $p < 1$, there is a positive probability for eigenvalues to collide in finite time (the ordering constraint is therefore useful at those points to restart). The idea is then to show that collisions are in a sense sufficiently rare for the above SDE to make sense (see [10,13] for further details). Using the SDE (10), one can derive as above the stationary distribution for the joint distribution of eigenvalues, which is still given by Eq. (3) but with now $\beta = \alpha = p \leq 1$. A very similar construction can be achieved in the Gaussian unitary ensembles case, leading to $\beta = 2p$. As announced, our dynamical procedure, that alternates standard and free addition of random matrices, can lead to any beta ensemble with $\beta \leq 2$. The corresponding matrices $\mathbf{M}(t)$ are furthermore invariant under the orthogonal (or unitary) group. This is intuitively clear, since both alternatives (adding a free slice or adding a commuting slice) respect this invariance and lead to a Haar probability measure for the eigenvectors (i.e., uniform over the orthogonal or unitary group). We have also proved that a collision leads to a complete randomization of the eigenvectors within the two-dimensional subspace corresponding to the colliding eigenvalues; see again [10].

It is well known that the eigenvalue density corresponding to the measure P^* given by Eq. (3) is the Wigner semicircle for any $\beta > 0$. In fact, using Eq. (5) with now $\alpha = \beta = p$, one immediately finds that the eigenvalue density is a semicircle with edges at $\pm\sqrt{2\beta N}$. We simulated numerically the matrix $\mathbf{M}_n(t)$ with $N = 200$ for a very small step $1/n$ and until a large value of t so as to reach the stationary distribution for the eigenvalues. Then we started recording the spectrum and the nearest neighbor spacings (NNS) every 100 steps so as to sample the ensemble. We verified that the spectral density of $\mathbf{M}_n(t = \infty)$ is indeed in very good agreement with the Wigner semicircle distribution for $\beta = 1/2$. Our sample histogram for the NNS distribution (NNSD) is displayed in Fig. 1. We also added the corresponding Wigner surmise (which is expected to provide a good approximate description of the NNSD).

From the point of view of a crossover between the standard Gaussian central limit theorem for random variables and the Wigner central limit theorem for random matrices, we see that as soon as the probability p for a noncommuting slice is positive, the asymptotic density is the Wigner semicircle, with a width of order $\sqrt{pN}$. A continuous crossover therefore takes place for $p = 2c/N$ with c strictly positive and independent of N. When $c = 0$, $\rho(\lambda)$ is a Gaussian of rms 1, which indeed corresponds to the solution of Eq. (5) for $\alpha = 0$. The SDE for the system $[\lambda_i(t)]$ becomes

$$d\lambda_i = -\frac{1}{2}\lambda_i dt + \frac{c}{N}\sum_{j\neq i}\frac{dt}{\lambda_i - \lambda_j} + db_i, \qquad (11)$$

with the additional ordering constraint $\lambda_1(t) \leq \cdots \leq \lambda_N(t)$ and the stationary joint PDF is still given by Eq. (3) but with now a vanishing repulsion coefficient $\beta = 2c/N$. In order to elicit the crossover, we study Eq. (5) with $\alpha = 2c/N$. The stationary differential equation corresponding to Eq. (5) (note this time that all terms are of the same order and the second derivative term is not negligible) can be integrated with respect to z again as

$$cG^2 + zG + \frac{dG}{dz} = -1, \qquad (12)$$

where the integration constant comes from the boundary condition $G \sim -1/z$ for $z \to \infty$. Note that Eq. (12) can be recovered directly from the saddle point equation route: under the measure P^* with $\beta = 2c/N$, the energy of a configuration of the λ_i's can be expressed in term of the continuous state density ρ, neglecting terms $\ll 1$, as

$$\mathcal{E}[\rho] = \frac{1}{2}\int \lambda^2\rho(\lambda)d\lambda - c\iint \ln(|\lambda - \lambda'|)\rho(\lambda)\rho(\lambda')d\lambda d\lambda'.$$

The probability density P^* therefore is rewritten in terms of ρ as

$$P^*[\rho] = Z\exp\left(-N\left[\mathcal{E}[\rho] + \int \rho\ln(\rho)\right]\right)\delta\left(\int\rho - 1\right),$$

where the entropy term, which is negligible when $\beta = p$ is of order 1, is now of the same order as the energy term (see [14] for a detailed discussion on the origin of the entropy

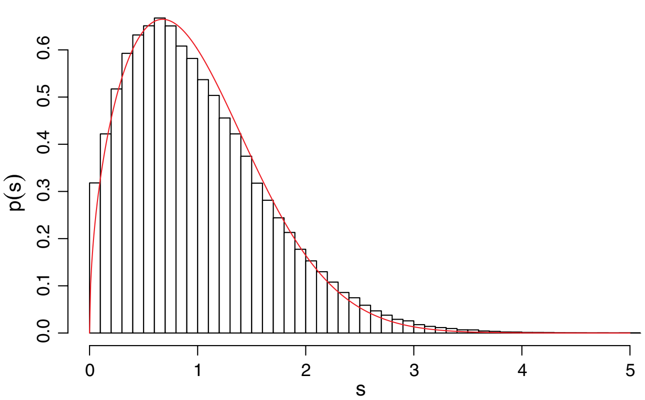

FIG. 1 (color online). Empirical NNSD $P(s)$ for the matrix $\mathbf{M}_n(t = \infty)$ for $\beta = p = 1/2$ with the Wigner surmise (red curve) corresponding to $\beta = \frac{1}{2}$, which behaves as s^β when $s \to 0$.

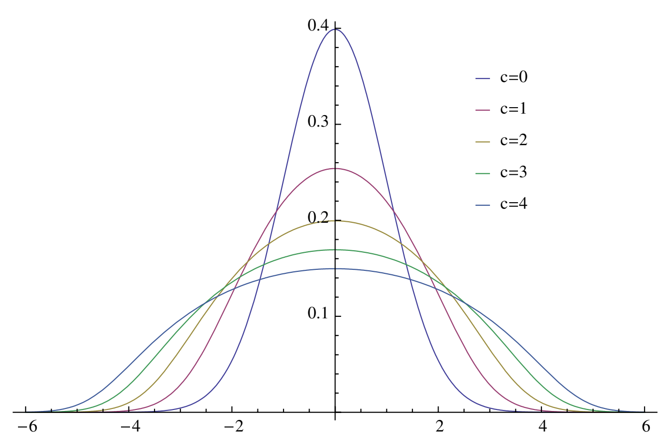

FIG. 2 (color online). Density $\rho_c(u)$ for $c = 0, 1, 2, 3, 4$ showing the progressive deformation of the Gaussian toward Wigner's semicircle. The value $c = 0$ corresponds to the highest curve at the origin, $c = 1$ to the second highest.

term). We now need to minimize the quantity $\mathcal{E}[\rho] + \int \rho \ln(\rho)$ with respect to ρ. It is easy to see that the unique minimizer ρ_c satisfies

$$\int \frac{\lambda \rho_c(\lambda)}{\lambda - z} d\lambda - 2c \iint \frac{\rho_c(\lambda)\rho_c(\lambda')}{(\lambda - z)(\lambda - \lambda')} d\lambda d\lambda' + \int \frac{\rho_c'(\lambda)}{\lambda - z} d\lambda + \nu = 0,$$

where ν is an integration constant. It is now straightforward to derive Eq. (12) from this last equation by identifying each term and choosing the constant ν so as to have the correct boundary condition for the Stieltjes transform of a probability measure. As expected physically, the diffusion term in Eq. (12) corresponds exactly to the entropy contribution to the saddle point.

Equation (12) was studied in detail by Askey and Wimp [15] and Kerov [16] (see also [17]). Set $G(z) := u'(z)/cu(z)$ to obtain a second order equation on u:

$$u''(z) + zu'(z) + cu(z) = 0. \tag{13}$$

It follows from the asymptotic behavior of $G(z)$ that, for $|z| \to \infty$,

$$u(z) \sim \frac{A_1}{z^c}. \tag{14}$$

Equation (13) can in turn be transformed with the change of function $u(z) := e^{-z^2/4} y(z)$ into a Schrödinger equation on $y(z)$:

$$y''(z) + [c - \tfrac{1}{2} - \tfrac{1}{4}z^2] y(z) = 0. \tag{15}$$

The solutions of Eq. (15) are known (see [18]) to write as $y(z) = A_2 D_{c-1}(z) + A_3 D_{-c}(iz)$, where D_{c-1}, D_{-c} are parabolic cylinder functions and where A_2 and A_3 are two constants. The general solution for u therefore is $u(z) = e^{-z^2/4}[A_2 D_{c-1}(z) + A_3 D_{-c}(iz)]$, and the correct asymptotic behavior of u is fulfilled for $A_2 = 0$. Now, one can recover the spectral density $\rho_c(\lambda)$ associated to G by the classical inversion formula and various elegant tricks [19]. The final result for $\rho(\lambda)$ reads, for all $c > 0$,

$$\rho_c(\lambda) = \frac{1}{\sqrt{2\pi}\Gamma(1+c)} \frac{1}{|D_{-c}(i\lambda)|^2}; \quad D_{-c}(z) = \frac{e^{-z^2/4}}{\Gamma(c)} \int_0^\infty dx e^{-zx-(x^2/2)} x^{c-1}. \tag{16}$$

Expression (16) was again checked with numerical simulations with very good agreement. The density ρ_c is represented for several values of c in Fig. 2. The integral representation for $D_{-c}(z)$ does not hold for $c = 0$, but the function $D_{-c}(iu)$ is still well defined for all $c \in (-1; 0]$ (see [15]). It is easy to check that $\rho_0(u) = e^{-u^2/2}/\sqrt{2\pi}$ when $c = 0$, as expected. When $c \to \infty$, the Wigner semicircle law is recovered:

$$\rho_c(u) \approx \frac{1}{2\pi c}\sqrt{4c - u^2}. \tag{17}$$

Standard results [18] on D_{-c} enable us to find the tails of ρ_c:

$$\rho_c(u) \sim u^{2c} e^{-u^2/2} \qquad (|u| \to \infty). \tag{18}$$

Let us return to Eq. (5) for $\beta = \alpha \in (0; 2)$. Interestingly, our method allows us to compute the correction to the Wigner semicircle inside the support of the spectral density for large but finite N due to the last diffusion term, which is usually neglected. Indeed, one can solve as above the stationary equation of Eq. (5) keeping every term. This leads to the following corrected spectral density, valid for large but finite N:

$$\rho(\lambda) = \frac{\sqrt{\alpha}}{\sqrt{2\pi}\Gamma(1+c)} \frac{1}{|D_{-c}(i\sqrt{\alpha}\lambda)|^2}, \tag{19}$$

where $\alpha = 2/(2-\beta)$ and $c = \beta N/(2-\beta)$. Note that this correction is valid only inside the spectrum and does not describe the edge scaling behavior nor the Tracy-Widom tails.

The above discussion can also be formally extended to $-1 \le c < 0$, corresponding to a weakly attracting Coulomb gas (also mentioned in [7]; see also [20] for an application). We conjecture that the stationary density for large system is again given by the above Askey-Wimp-Kerov distributions ρ_c but for the parameter range $c \in (-1; 0]$. For $c = -1$, the stationary density ρ_{-1} is a Dirac mass at 0. Beyond this level, the attraction is too strong, and the gas completely collapses on itself.

As a conclusion, we have provided here the first explicit construction of invariant beta ensembles of random matrices, for arbitrary $\beta \le 2$. The stationary distribution for the eigenvectors is the Haar probability measure on the orthogonal group if $0 < \beta \le 1$, respectively unitary group if $1 < \beta \le 2$. We have found a natural scaling limit that allows one to interpolate smoothly between the Gaussian distribution, relevant for sums of independent random variables, and the Wigner semicircle distribution, relevant for sums of free random matrices. The interpolating limit distributions form a one parameter family that can be explicitly computed. The statistics of the largest eigenvalue is also very interesting (and now well known for $\beta > 0$; see [21–24]): one should be able to interpolate smoothly, as a function of c, between the well-known Gumbel distribution of extreme value statistics and the Tracy-Widom (β) distributions. Whether this can be mapped into a generalized Kardar-Parisi-Zhang or directed polymer problem remains to be seen.

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