

Sub-Binomial Light

J. Sperling* and W. Vogel

Arbeitsgruppe Quantenoptik, Institut für Physik, Universität Rostock, D-18051 Rostock, Germany

G. S. Agarwal

Department of Physics, Oklahoma State University, Stillwater, Oklahoma 74078, USA

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The click statistics from on-off detector systems is quite different from the counting statistics of the more traditional detectors. This necessitates the introduction of new parameters to characterize the nonclassicality of fields from measurements using on-off detectors. To properly replace the Mandel Q_M parameter, we introduce a parameter Q_B . A negative value represents a sub-binomial statistics. This is possible only for quantum fields, even for super-Poisson light. It eliminates the problems encountered in discerning nonclassicality using Mandel's Q_M for on-off data.

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Introduction.—Nonclassicality of the radiation fields has been at the heart of quantum optics. One uses detectors which work by absorption of photons and hence normally ordered correlations are the ones measured directly [1–3]. Nonclassicality in quantum optics has therefore been formulated in terms of the nonclassical properties of the P function associated with the density matrix of the quantum fields [4,5]. However, the P function itself is not directly measurable. Mandel introduced an experimentally deducible measure of nonclassicality, namely the Q_M parameter, defined by

$$Q_M = \frac{\langle(\Delta n)^2\rangle}{\langle n\rangle} - 1, \quad (1)$$

with $\langle n\rangle$ and $\langle(\Delta n)^2\rangle$ being the classical mean value and the classical variance of the photoelectric statistics [6]. If Q_M is negative, then the photocounting statistics is of sub-Poisson type and we conclude that the field is nonclassical. The very first experimental demonstration of this nonclassical effect was given in Ref. [7].

More recently, it has become necessary to use photon number resolving (PNR) detectors to discriminate between states with definite photon numbers [8–11]. Because such detectors are not directly available, one uses on-off detector systems (avalanche photodiodes) [12]. Such detector systems have been characterized by tomographic methods [13]. The deduction of nonclassicality using measurements with avalanche photodiodes and using the Mandel Q_M parameter meets with difficulties. For example, even if the field is completely classical, then Q_M can be negative [14].

In this Letter, we present a solution to this difficulty by introducing an appropriate measure of nonclassicality using the data from on-off detector systems. The condition $Q_B < 0$ characterizes the sub-binomial click statistics of light. We provide a physical justification for the new measure and we show by several examples the validity of

the binomial Q_B parameter. Whenever the discrimination of adjacent photon numbers is of relevance for applications in modern quantum technologies, the notion of sub-binomial light is expected to play a vital role.

The binomial Q_B parameter.—The traditional detectors work on the principle that a photoelectron is emitted if a photon is absorbed. Perturbation theory shows that the emission probability is proportional to the intensity of light and this leads to the counting distribution [6,15],

$$p_n = \left\langle : \frac{(\eta \hat{n} + \nu)^n}{n!} e^{-(\eta \hat{n} + \nu)} : \right\rangle. \quad (2)$$

Herein, the operator \hat{n} represents the photon number, η the detection efficiency, ν the number of noise or dark counts, and the $: \cdot :$ notation indicates the normal ordering prescription. For on-off detectors the mechanism is different. The detector clicks for any number of photons and does not click if the field is in the vacuum state. We showed recently that if one employs N on-off detectors, then the counting distribution is given by [14]

$$c_k = \left\langle : \frac{N!}{k!(N-k)!} (e^{-(\eta \hat{n} + \nu)/N})^{N-k} (\hat{1} - e^{-(\eta \hat{n} + \nu)/N})^k : \right\rangle. \quad (3)$$

Here ν is the noise count of the whole PNR detector system, whereas in Ref. [14] we considered the noise counts of an individual on-off detector. We also observed, that the counting or click statistics converges to the true statistics with increase in the number of detectors. However this convergence is slow as it goes as $1/N$. For coherent states, Eqs. (2) and (3) reduce to the Poisson and binomial statistics, respectively.

Because the counting distribution has a different form, one would expect that one needs a measure different from the Mandel Q_M parameter to characterize nonclassicality. Note that the traditional photocounting distribution

involves the expectation of a normally ordered Poisson distribution, whereas the click statistics involves a normally ordered binomial one. We expect that an appropriate measure of the nonclassical statistics would be the *sub-binomiality* of the distribution. Hence we introduce Q_B defined by

$$Q_B = N \frac{\langle(\Delta c)^2\rangle}{\langle c\rangle(N - \langle c\rangle)} - 1, \quad (4)$$

where $\langle c\rangle$ is the mean number of clicks, and $\langle(\Delta c)^2\rangle$ is the variance of the click statistics $(c_k)_{k=0}^N$,

$$\langle c\rangle = \sum_{k=0}^N k c_k \quad \text{and} \quad \langle(\Delta c)^2\rangle = \sum_{k=0}^N (k - \langle c\rangle)^2 c_k. \quad (5)$$

The moments $\langle c\rangle$ and $\langle(\Delta c)^2\rangle$ are defined the sense of classical probabilistic quantities.

This Q_B possesses the following properties: (1) The Q_B parameter must not yield negative values for classical states; (2) For any quantum state having a binomial counting statistics $(c_k)_{k=0}^N$, Q_B should be zero; (3) It is based on first and second moments of $(c_k)_{k=0}^N$; (4) For $N \rightarrow \infty$, Q_B should converge to Q_M .

The definition of Q_B requires at least two on-off detectors. A single on-off detector only yields one click and no click with probability $c_0 = p$ and $c_1 = 1 - p$ ($0 \leq p \leq 1$), respectively. Thus, any quantum state has a binomial statistics, $Q_B = 0$, as long as a single detector is used.

We next prove that $Q_B < 0$ is a measure of nonclassicality for measurements with N on-off detectors. For this purpose, it is convenient to use the generating function of the click statistics

$$f(x) = \sum_{k=0}^N c_k x^k = \langle: [x(\hat{1} - e^{-(\eta\hat{n}+\nu)/N}) + e^{-(\eta\hat{n}+\nu)/N}]^N : \rangle. \quad (6)$$

From the derivatives of f , one can obtain all moments of the statistics. It can be shown that the variance $\langle(\Delta c)^2\rangle$ reads as

$$\langle(\Delta c)^2\rangle = N(N-1) \langle: (\Delta e^{-(\eta\hat{n}+\nu)/N})^2 : \rangle + N \left(1 - \frac{\langle c\rangle}{N}\right) \frac{\langle c\rangle}{N}, \quad (7)$$

with $\langle c\rangle/N = 1 - \langle: \exp[-(\eta\hat{n} + \nu)/N] : \rangle$, cf. [16]. Rewriting this equation according to Eq. (4),

$$Q_B = \frac{(N-1) \langle: (\Delta e^{-(\eta\hat{n}+\nu)/N})^2 : \rangle}{\langle: e^{-(\eta\hat{n}+\nu)/N} : \rangle (1 - \langle: e^{-(\eta\hat{n}+\nu)/N} : \rangle)}, \quad (8)$$

we obtain the binomial Q_B parameter in its explicit form. Note that the parameter Q_B , as it is clearly seen from this result, depends on higher-order moments of the photon number statistics, which is not the case for Mandel Q_M parameter given in Eq. (1). Such higher-order moments are

beginning to be studied in experiments [17,18]. For a classical state, when the P function has the properties of a classical probability distribution [19], it yields that any normally ordered variance is non-negative. In addition, $\langle c\rangle/N$ and $1 - \langle c\rangle/N$ are non-negative mean values. It follows for classical states

$$Q_B \geq 0. \quad (9)$$

Let us note that the individual expectation values are independent of the phase. Altogether, this proves the claim that a negative binomial Q_B value implies a nonclassical photon statistics.

Concerning the convergence properties of Q_B , we can use the following result. In Ref. [14], we have already shown that the click statistics converges to the photo statistics for $N \rightarrow \infty$. It follows that $\langle(\Delta c)^2\rangle/\langle c\rangle$ converges to $\langle(\Delta n)^2\rangle/\langle n\rangle$. The only difference left between Q_M and Q_B , cf. Eqs. (1) and (4), is

$$\frac{N}{N - \langle c\rangle} = \frac{1}{1 - \frac{\langle c\rangle}{N}}. \quad (10)$$

Because $\langle c\rangle$ converges to the finite value of $\langle n\rangle$, we obtain

$$Q_B \rightarrow Q_M \quad \text{for } N \rightarrow \infty. \quad (11)$$

As a last property we verify that for coherent states $Q_B = 0$. According to Eq. (3), $(c_k)_{k=1}^N$ is a binomial distribution with

$$\langle c\rangle = N(1 - e^{-(\eta|\alpha|^2 + \nu)/N}), \quad (12)$$

$$\langle(\Delta c)^2\rangle = N(1 - e^{-(\eta|\alpha|^2 + \nu)/N})e^{-(\eta|\alpha|^2 + \nu)/N}.$$

Applying the binomial Q_B parameter, we obtain the desired interpretation, $Q_B = 0$. The parameter Q_B does not lead to fake nonclassicality, for any choice of noise or detection efficiency. For more general states having a binomial statistics, we can formulate similarly $Q_B = 0$.

Our binomial Q_B parameter is directly constructed for measurements with on-off detector systems, including imperfections. It can discern nonclassicality in experiments using only two or more on-off detectors. In the following, we apply the Q_B parameter to typical examples in quantum optics. We consider three kinds of states having comparable mean photon numbers. Usually the main source of imperfections is caused by the quantum efficiency $\eta < 1$, so that we may assume a negligible noise count rate, $\nu \approx 0$.

Thermal states.—First, we may consider a classical, thermal state, with a mean photon number $\langle n\rangle = \bar{n}$ and a variance $\langle(\Delta n)^2\rangle = \bar{n}(\bar{n} + 1)$. Such a state enables us to highlight the difficulty associated with the value of Q_M for data from on-off detectors. It has a positive Mandel parameter, $Q_M = \bar{n}$. The efficiency simply scales the mean photon number \bar{n} to a smaller value, $\eta\bar{n}$.

Using the P function of the thermal state, we obtain the click statistics in the form of a beta-binomial distribution $c_k = f_{N,\alpha,\beta}(k)$ for $\alpha = 1$ and $\beta = N/\bar{n}$,

$$c_k = \frac{N!}{k!(N-k)!} \frac{\Gamma(N-k+\frac{N}{\bar{n}})\Gamma(k+1)}{\Gamma(N+1+\frac{N}{\bar{n}})} \frac{N}{\bar{n}}, \quad (13)$$

see Supplemental Material [16]. The well-known mean values and variances of such distributions yield Q_B values,

$$Q_B = \frac{N + \frac{N}{\bar{n}} + 1}{\frac{N}{\bar{n}} + 2} - 1 = \frac{N-1}{\frac{N}{\bar{n}} + 2}. \quad (14)$$

In Fig. 1, we plotted different parameters depending on the number N of on-off detectors, with $2 \leq N \leq 16$. Determining the Mandel parameter value from the click statistics—denoted as Q_F —leads to fake nonclassicality. The binomial parameter, $Q_B > 0$, correctly displays the classicality of thermal light. For large numbers of on-off detectors, Q_B approaches the value of $Q_M = \bar{n}$.

Fock states.—Second, we may study a Fock state for m photons. To solely consider the effects of on-off detectors, we choose the detection efficiency $\eta = 1$. The photon statistics of the Fock state is a singular one, $p_n = \delta_{n,m}$ and $Q_M = -1$.

To obtain the click statistics from the true photo statistics, we apply Eq. (14) of Ref. [14] to get

$$c_k = \sum_{n=0}^{\infty} \frac{N!}{k!(N-k)!} \frac{\partial_y^n (e^y - 1)^k |_{y=0}}{N^n} p_n. \quad (15)$$

The particular example of a Fock state yields

$$c_k = \frac{N!}{k!(N-k)!} \frac{\partial_y^m (e^y - 1)^k |_{y=0}}{N^m}. \quad (16)$$

The mean click number $\langle c \rangle$ and the variance $\langle (\Delta c)^2 \rangle$ are given by

$$\langle c \rangle = N \left(1 - \left[1 - \frac{1}{N} \right]^m \right), \quad (17)$$

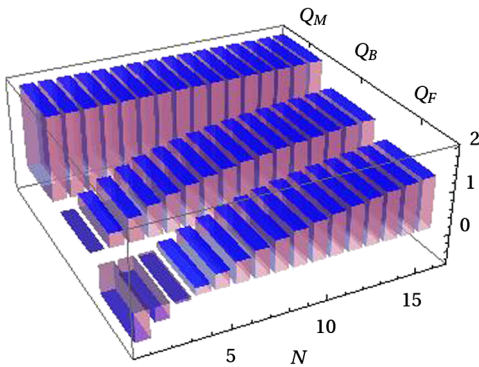


FIG. 1 (color online). Different Q parameters are plotted for a classical, thermal state, with $\langle n \rangle = \eta \bar{n} = 2$ and $\nu = 0$. The true Mandel parameter is Q_M , the binomial one is Q_B . Using the definition in Mandel's form for click statistics yields Q_F .

$$\langle (\Delta c)^2 \rangle = N(N-1) \left(1 - 2 \left[1 - \frac{1}{N} \right]^m + \left[1 - \frac{2}{N} \right]^m \right) + \langle c \rangle - \langle c \rangle^2, \quad (18)$$

cf. [16]. Thus, the analytical expression of the Q_B parameter for m photons is

$$Q_B = (N-1) \frac{N^m (N-2)^m - (N-1)^{2m}}{[N^m - (N-1)^m](N-1)^m}. \quad (19)$$

In Fig. 2, we plotted the Q_B parameter depending on the number of photons, m , and the number of available on-off detectors, N . The verification of a nonclassical photon number statistics can be directly observed from $Q_B < 0$. This is possible, although the considered detector system is unable to measure the true photo statistics. It is also clear, that a larger number of photons m requires a higher number of detectors to significantly identify nonclassicality. Surprisingly, a measurement using only two on-off detectors can be used to infer nonclassical light. We can also observe that for large numbers of on-off detectors Q_B approaches the value $Q_M = -1$.

Single-photon-added thermal state.—In this last example, we show that the binomial Q_B parameter can detect nonclassical photon statistics beyond sub-Poisson ones. For this purpose let us study a single-photon-added thermal state (SPATS) [20]. This state has been experimentally realized [21], and its nonclassicality has been verified by reconstructing its P function [22]. The P function of the SPATS is given by

$$P_{\text{SPATS}}(\alpha) = \frac{1}{\pi \bar{n}^3} [(1 + \bar{n})|\alpha|^2 - \bar{n}] e^{-(|\alpha|^2/\bar{n})}. \quad (20)$$

A straightforward computation of Q_M yields for the efficiency η

$$Q_M = \eta \frac{\bar{n}^2 - \frac{1}{2}}{\bar{n} + \frac{1}{2}} \begin{cases} \geq 0 & \text{for } \bar{n} \geq \sqrt{0.5}, \\ < 0 & \text{for } \bar{n} < \sqrt{0.5}. \end{cases} \quad (21)$$

For $\bar{n} > \sqrt{1/2}$, the Q_M parameter cannot identify the nonclassicality of the SPATS.

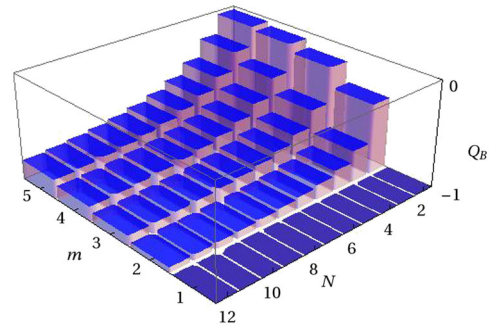


FIG. 2 (color online). The plot shows the binomial Q_B parameter for the click statistics measured by N on-off detectors, with $2 \leq N \leq 12$. The computed example shows the sub-binomial statistics of m photon Fock states ($1 \leq m \leq 5$).

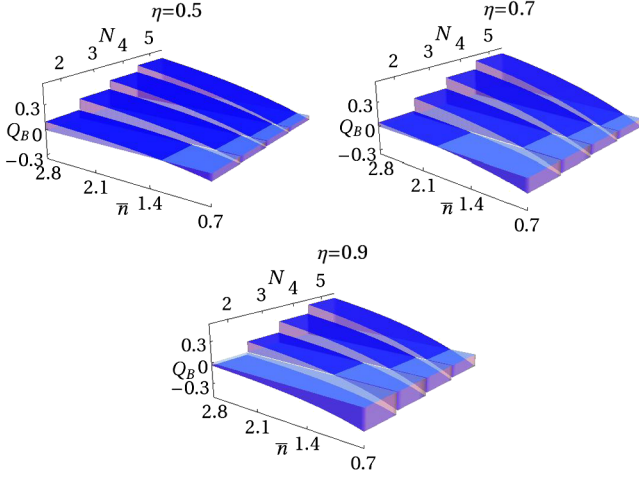


FIG. 3 (color online). The plot shows the binomial Q_B parameter for the counting statistics measured by a PNR detector with $N = 2, 3, 4, 5$ on-off detectors. The computed example is a SPATS with a mean thermal photon number $\sqrt{0.5} \leq \bar{n} \leq 4\sqrt{0.5}$. The individual plots have a quantum efficiency $\eta = 0.5, 0.7, 0.9$.

In the following we apply the Q_B parameter to the SPATS. Some algebra, using the P_{SPATS} distribution, yields

$$Q_B = (N - 1) \frac{I(\frac{2\eta}{N}) - I(\frac{\eta}{N})^2}{I(\frac{\eta}{N})[1 - I(\frac{\eta}{N})]}, \quad (22)$$

the needed integral $I(\lambda)$ can be analytically computed [16]. In Fig. 3, we plotted the binomial Q_B parameter depending on \bar{n} . For the chosen parameters we have a super-Poisson statistics, $Q_M \geq 0$. However, we find regions with a sub-binomial statistics, $Q_B < 0$. We obtain the surprising result that the click statistics can be more suitable to detect nonclassicality than the Q_M parameter, even for a small number of on-off detectors.

Summary and conclusions.—We established the binomial Q_B parameter. It serves for the identification of nonclassical radiation measured with multiple on-off detectors including imperfections. A negative parameter, $Q_B < 0$, refers to sub-binomial light. We showed that the binomial Q_B parameter convergences to the original Mandel Q_M parameter for large numbers of on-off detectors. It is worth mentioning that our method does not require a reconstruction of the true photon number statistics.

We applied our method to typical states, for example, Fock states representing sub-binomial light. We also studied the statistics of a SPATS measured by only a small number of on-off detectors. In this case, the Q_B parameter can identify nonclassical, in particular sub-binomial, states of light beyond the Q_M parameter. From a more general

perspective, sub-binomial light and its characterization may be of vital relevance whenever modern quantum technologies require the discrimination of photon number states.

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*jan.sperling2@uni-rostock.de

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