P H Y S I C A L   R E V I E W   L E T T E R S

Anomalous Chiral Fermi Surfaces

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We provide a geometrical argument for the emergence of a Wess-Zumino-Witten (WZW) term in a Fermi surface threaded by a Berry curvature. In the presence of external fields, the gauged WZW term yields a chiral (triangle) anomaly for the fermionic current at the edge of the Fermi surface. The fermion number is conserved, though, since Berry curvatures always occur in pairs with opposite (monopole) charge. The anomalous vector and axial currents for a fermionic fluid at low temperature threaded by pairs of Berry curvatures are discussed. The leading temperature correction to the chiral vortical effect in a slowly rotating Fermi surface threaded by a Berry curvature may be tied to the gravitational anomaly.

Introduction.—Chiral anomalies play a key role in QCD at low energy where for instance the chiral triangle anomaly causes the $\pi^0 \to \gamma\gamma$ decay. Chiral anomalies have a geometrical origin that is concisely captured by Wess-Zumino-Witten (WZW) terms [1,2]. Anomalies in fermionic systems at finite density were noted in Ref. [3]. Similar anomalies have emerged in dense holography [4,5] which have a general explanation from entropy constrained hydrodynamics [6], and more generically WZW effective actions with matter [7,8]. The role of the WZW term in chiral fluids including the role of the gravitational anomaly, was recently analyzed in Ref. [9]. Berry phases in fermionic systems are intricately related to WZW-like terms. Pertinent examples can be found in the chiral bag models of hadrons [10].

Recently, a triangle anomaly was argued to take place in Fermi surfaces with Berry curvatures [11], a point of relevance to Weyl semimetals [12] and possibly graphene [13]. A careful but involved analysis of the phase space structure of Fermi liquids in the presence of a Berry curvature has revealed a triangle anomaly in the fermionic current that tags on the monopole charge of the Berry phase. Since Berry curvatures always occur in dual pairs, charge conservation is of course garenteed. The interplay between a Berry phase and the chiral anomaly in QCD was originally noted in Ref. [14]. The relevance of the chiral anomaly to condensed matter physics with numerous applications to ^{3}He-*A* was also emphasized in Ref. [15] (and references therein).

In this Letter, we show that the triangle anomaly in the fermionic current follows from a WZW-like contribution of the Berry curvature after a homotopic extension of the quasiparticle adiabatic momentum and a minimal $U(1)$ gauging. As expected, the normalization of the WZW-like action follows solely from geometry once the monopole charge is given. The ensuing currents at the edge of the Fermi surface are plagued by the chiral (triangle) anomaly. At low temperature, a slowly rotating Fermi surface threaded by Berry curvatures exhibit both chiral

magnetic and vortical effects in the presence of external $U(1)$ fields [16]. The leading temperature correction to the chiral vortical effect maybe traced back to the gravitational anomaly [9,15,17].

Berry curvature.—The emergence of a Berry phase at the Fermi surface results from the occurence of a level crossing between a particle and a hole state. This crossing is dynamical in origin and maybe due to an accidental zero in the momentum dependent gap function for weakly coupled BCS metals or some intricate dynamics in the particle level dispersion law as in Weyl semimetals or graphene for instance. The particles and holes (quasiparticles) near the zero act as slow variables in the presence of an induced Berry phase generated by the rest of the Fermi surface acting as fast variables. Near the zero or crossing point, the fermion dispersion relation is linear causing the fermions to behave nearly relativistically. As a result, the dynamics near the Fermi surface is effectively $1 + 1$ dimensional making the setup ideal for the emergence of chiral anomalies and, potentially, the general lore of anomalous bosonization.

For simplicity, consider a level crossing at the edge of a Fermi surface with canonical degeneracy 2, making the induced Berry curvature a monopole of charge $g = 1/2$ (see below). If we denote by $\vec{\mathcal{A}}(\vec{p})$, the pertinent Berry phase or curvature near such a zero or level crossing, then it couples to a particle or a hole with (adiabatic) momentum $\vec{p}(t)$ through

$$\mathbf{S}_B = \int dt \vec{\mathcal{A}}(\vec{p}) \cdot \dot{\vec{p}}. \tag{1}$$

Since the Berry curvature is monopolelike and Abelian, it is threaded by a Dirac string and the action (1) in $0 + 1$ dimensions is nonlocal. It can be made local by extending it homotopically to $1 + 1$ dimensions following the original arguments in Ref. [2]. For that we define

$$\vec{p}(t) \to k_F \hat{p}(t, s) \tag{2}$$

with k_F the Fermi momentum, $\hat{p}(t, 0) = \hat{e}$ some fixed unit vector and $\hat{p}(t, 1) = \hat{p}$. The manifold $\mathcal{D} = S^1 \times [0, 1]$ can be regarded as the upper cup of S^2 with $\bar{\mathcal{D}}$ its complement,

i.e., $S^2 = \mathcal{D} + \bar{\mathcal{D}}$. While $\Pi_{0+1}(S^2) = 0$, $\Pi_{1+1}(S^2) = \mathbf{Z}$. The latter is characterized by the topological charge or winding density

$$\mathbf{W}(\hat{p}) = \frac{1}{8\pi} \epsilon^{ij} \epsilon^{abc} \hat{p}^a \partial_i \hat{p}^b \partial_j \hat{p}^b \qquad (3)$$

with the labeling i, $j = 0, 5$, and a, b, $c = 1, 2, 3$, the coordination $x_0 = t$ and $x_5 = s$. Equation (3) carries unit normalization

$$\int_{S^2} d^2x \mathbf{W}(\hat{p}) = \frac{1}{4\pi} \int_{S^2} \hat{p} \cdot (d\hat{p} \times d\hat{p}) = 1. \qquad (4)$$

The Berry curvature in (1) is a monopole flux

$$\mathbf{S}_B = \int_{\mathcal{C} = \partial \mathcal{D}} \vec{\mathcal{A}}(\vec{p}) \cdot d\vec{p} \equiv \int_{\mathcal{D}} d\mathcal{A}(\vec{p}) \qquad (5)$$

with the form notation subsumed. A Berry curvature of unit flux through the Fermi surface amounts to

$$\int_{S^2 = \mathcal{D} + \bar{\mathcal{D}}} d\mathcal{A} = g \int_{S^2} \hat{p} \cdot (d\hat{p} \times d\hat{p}) = 2\pi, \qquad (6)$$

which is effectively a hedgehog monopole in momentum space of charge $g = 1/2$ centered in the Fermi sea. Therefore, the nonlocal form (1) in $0 + 1$ dimensions can be rewritten in local form in $1 + 1$ dimensions through

$$\mathbf{S}_B = 4\pi g \int_{\mathcal{D}} d^2x \mathbf{W}(\hat{p})$$
$$= \frac{g}{2k_F^3} \int_{\mathcal{D}} d^2x \epsilon^{ij} \epsilon^{abc} p^a \partial_i p^b \partial_j p^b \qquad (7)$$

after relaxing the form notation. Equation (7) is the WZW term for the Berry curvature of flux 1 or monopole charge $g = 1/2$ centered in a Fermi sphere of radius k_F; it is also well established in condensed matter [15,18,19].

A Berry curvature of flux k through the Fermi surface amounts to a monopole of charge $g = k/2$ centered in the Fermi sphere, while a Berry curvature of flux $-k$ amounts to an antimonopole of charge $g = -k/2$. A monopole will cause the particle and hole excitations at the zero or level crossing to be right-handed, say, while an antimonopole will cause them to left-handed. The labelling right and left is conventional. In materials level crossings occur always in pairs making the net flux always zero.

Chiral anomaly.—To assess the effect of the Berry curvature (7) on the transport of the fermion number current around the zero or level crossing in the presence of external electromagnetism and at the edge of the Fermi surface, we need to gauge (7). For that we note that if we were to formally extend

$$p^a(t, s) \rightarrow p^\mu(t, s, \vec{x}) \qquad (8)$$

for the sake of the argument [we will revert to $p^a(t, s)$ shortly], which is seen to be valued in $\mathcal{D} \times R^3$, then (7) turns to a Chern Simons—like contribution in five dimensions

$$\mathbf{S}_{\perp B} \equiv N_\perp \mathbf{S}_B = \frac{g}{2k_F^3} \frac{N_\perp}{V_3} \int_{\mathcal{D} \times R^3} p(dp)^2 \qquad (9)$$

for $N_\perp$ quasiparticles. Here $N_\perp/V_3 = n_\perp k_F$ and $n_\perp = k_F^2/2\pi^2$ denotes the fermion density at the Fermi surface for a single fermion species. Near a zero or level crossing, the quasiparticles contribute coherently to the Chern Simons—like term, thus the multiplication by $N_\perp$. Their number density is given by the density $n_\perp$ at the Fermi surface.

It is now straightforward to gauge (9), say by minimal substitution $p \rightarrow p + A$. Thus, the gauged WZW term in form notation is

$$\mathbf{S}_{\perp B} = \frac{g n_\perp}{2k_F^2} \int_{\mathcal{D} \times R^3} (p + A)\left(dp + \frac{1}{2}F\right)^2. \qquad (10)$$

We now revert to $p^\mu \rightarrow p^a(t, s)$ for our Berry curvature or hedgehog monopole. F is the $U(1)$ field strength. From (10) it follows that the Fermionic current is anomalous. Indeed, in the presence of a $U(1)$ gauge field A, the fermionic current carried by the particle and hole excitations at the level crossing or zero can be thought of as either right-handed (Berry curvature with net positive flux) or left-handed (Berry curvature with net negative flux). Both currents couple normally to the $U(1)$ gauge field through

$$\mathbf{S}_{R,L} = \int_{R^4} \mathbf{J}_{R,L} A. \qquad (11)$$

The Noether construction $A \rightarrow A + d\xi$ shows that

$$d\mathbf{J}_R = \frac{g n_\perp}{2k_F^2}\left(dp + \frac{1}{2}F\right)^2 \qquad (12)$$

is anomalous. A similar relation holds for $\mathbf{J}_L$ with $g \rightarrow -g$. In (12) $p^\mu \equiv p^a(t, s = 1)$ [after reverting to $p^a(t, s)$], and its contribution has no support by the antisymmetric contraction. Thus

$$d\mathbf{J}_R = \frac{g n_\perp}{8k_F^2} F^2 \equiv \frac{g}{2\pi^2} \vec{E} \cdot \vec{B} \qquad (13)$$

after using the explicit contribution of the single species fermions at the Fermi surface. For the Berry curvature of general charge $g = k/2$ or k-fluxes through the Fermi surface, this is the result established recently in Ref. [11] using Fermi liquid theory and transport arguments and earlier in Ref. [15] in the context of ^{3}He-A.

Since at the zero or level crossing the particle-hole excitations are effectively Weyl, (13) is the expected anomaly for a free Weyl fermion in $1 + 3$ dimensions. Note that the features of the Fermi surface dropped out of (13) as expected from geometry. Anomalies are infrared manifestations of ultraviolet physics that are insensitive to matter [20].

In many ways, (13) carries the essentials of the anomaly matching condition in matter whereby the infrared degrees of freedom at the Fermi surface are mapped onto the ultraviolet character of the chiral anomaly in the vaccum.

It is worth noting that when the number of quasiparticles in (9) encompasses all of the Fermi surface, then $N_\perp/V_3 = (n_\perp k_F)/3$ is just the Fermi density, in which case

$$d\mathbf{J}_R = \frac{g n_\perp}{24 k_F^2} F^2 \equiv \frac{g}{6\pi^2} \vec{E} \cdot \vec{B}. \qquad (14)$$

Equations (13) and (14) are the covariant and consistent form of the Abelian chiral anomaly, respectively, both of which are known to follow from specific UV regularizations for fermions in the vacuum [21]. Here, they are realized through a different counting of states either at the surface (13) or through the bulk (14) of the Fermi surface. Both may have a realization in materials differing by the degree of coupling of the quasiparticles.

Anomalous fermionic fluid.—If we were to treat the quasiparticle excitations at the Fermi surface as a fluid with a (momentum-dependent) fluid velocity v^μ and a chemical potential μ_R (following the right convention for a zero or crossing with positive flux), then chiral magnetic and vortical contributions would be expected.

In the presence of anomalies the chemical potentials are defined using the conserved but gauge fixed right and left currents. We note that the gauge fixing in the vector current is redundant since it is conserved (see below). In leading order they follow readily by the substitution $A \to A + \mu_{R,L} v$ in the WZW term or the anomaly contribution [7]. Specifically, (13) now reads (for the right current)

$$\begin{aligned} d\mathbf{j}_R &= \frac{g}{16\pi^2}(F + 2\mu_R dv)^2 \\ &= \frac{g}{16\pi^2}(F^2 + 4\mu_R F dv + 4\mu_R^2 (dv)^2) \end{aligned} \qquad (15)$$

and $\mathbf{j}_R = \mathbf{n}_R v$ now the normal constitutive current with quasiparticle density $\mathbf{n_R}$. Equation (15) can be reshuffled in the form

$$d\tilde{\mathbf{j}}_R \equiv d\left(\mathbf{j}_R - \frac{g}{4\pi^2}(\mu_R F v + \mu_R^2 v dv)\right) = \frac{g}{16\pi^2} F^2. \qquad (16)$$

Equation (16) shows that the constitutive but normal current $\mathbf{j}_R$ acquires an anomalous contribution, the sum of which yields

$$\tilde{\mathbf{j}}_R \equiv \mathbf{j}_R - \frac{g}{4\pi^2}(\mu_R F v + \mu_R^2 v dv), \qquad (17)$$

which obeys the triangle or chiral anomaly on the Fermi surface. Similar relations hold for the left current with the substitution $g \to -g$. The vector current $\tilde{\mathbf{j}} = \tilde{\mathbf{j}}_R + \tilde{\mathbf{j}}_L$ is anomaly free, while the axial current $\tilde{\mathbf{j}}_A = \tilde{\mathbf{j}}_R - \tilde{\mathbf{j}}_L$ is anomalous.

The first anomalous contribution in (17) is the chiral magnetic contribution while the second anomalous contribution is the chiral vortical effect. For instance the spatial vector and axial currents flowing through a rotating but cold Fermi surface threaded by a dual pair of Berry curvatures read

$$\int \frac{d\hat{p}}{4\pi} \tilde{\mathbf{j}}^i = -\frac{g}{2\pi^2}(\mu_R - \mu_L)B^i + \frac{g}{4\pi^2}(\mu_R^2 - \mu_L^2)\omega^i$$

$$\int \frac{d\hat{p}}{4\pi} \tilde{\mathbf{j}}_A^i = -\frac{g}{2\pi^2}(\mu_R + \mu_L)B^i + \frac{g}{4\pi^2}(\mu_R^2 + \mu_L^2)\omega^i \qquad (18)$$

with $\vec{\omega}$ the external circular velocity. The emergence of a current in the presence of a slowly rotating Fermi surface was also noted in Refs. [22–24].

At low temperature the axial vortical effect is expected to be shifted

$$\mu_{R,L}^2 \to \mu_{R,L}^2 + \frac{(\pi T)^2}{24}, \qquad (19)$$

while the vector vortical effect is not. The temperature shift appears naturally in the context of a rotating superfluid $^3\text{He} - A$ system at low temperature and reflects on the generic character of the mixing between axial and gravitational anomalies in gapless constitutive systems [9,15,17]. A more microscopic description of the constitutive currents shows that the temperature shift follows from the leading tadpole corrections as discussed in Ref. [7] through thermal phonons at low temperature in a superfluid state, whereby $\Pi^2/F^2 \approx (\pi T)^2/\mu^2$ with $F^2 \approx n_\perp$ at the Fermi surface.

Conclusions.—We have shown that the triangle anomaly established recently in Ref. [11] follows from the pertinent gauged WZW term associated to the Berry curvature in a Fermi surface. This result was expected, as all anomalies are of geometrical nature and insensitive to the details of the underlying dynamics here taking place at the Fermi surface. The origin of the Berry curvature in a Fermi surface, while intricate dynamically, is manifested by particle-hole crossing at specific points of the Fermi surface. Near these points, the approximate quasiparticle spectrum is effectively two-dimensional and relativistic.

In the presence of $U(1)$ gauge fields, the right and left fermionic quasiparticle currents are anomalous with the chiral or triangle anomaly being the lore. Our analysis through (13) shows how the infrared degrees of freedom at the Fermi surface are mapped onto the ultraviolet content of the chiral anomaly. This is an example of how the anomaly matching condition operates around a Fermi surface. This point is of interest to dense QCD at weak coupling whereby Fermi surfaces are expected.

A rotating Fermi fluid threaded by Berry curvatures at low temperature exhibits both chiral and vortical effects. The leading temperature effects appear to be related to the gravitational anomaly noted in Refs. [9,15,17] and, perhaps, generic. These leading temperature effects are tadpolelike and unambiguous in the anomalous superfluid or effective Lagrangian analysis in Ref. [7]. They maybe measurable through the axial vortical effect.

Finally, it would be interesting to explore the possible occurrence of non-Abelian Berry curvatures in Fermi

surfaces, whereby non-Abelian anomalies can be realized. We recall that non-Abelian Berry phases emerge naturally in chiral bag models where three quarks are (somehow) trapped and contribute essentially to their quantum numbers [10]. Non-Abelianity requires additional internal degeneracies on the particle and hole quasiparticles at the edge of the Fermi surface with spin being an obvious candidate.

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[1] J. Wess and B. Zumino, Phys. Lett. B **37**, 95 (1971).

[2] E. Witten, Nucl. Phys. **B223**, 422 (1983).

[3] D. K. Hong, M. Rho, and I. Zahed, Phys. Lett. B **468**, 261 (1999); M. A. Nowak, M. Rho, A. Wirzba, and I. Zahed, Phys. Lett. B **497**, 85 (2001); D. T. Son and A. R. Zhitnitsky, Phys. Rev. D **70**, 074018 (2004).

[4] J. Erdmenger, M. Haack, M. Kaminski, and A. Yarom, J. High Energy Phys. 01 (2009) 055; N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam, and P. Surowka, arXiv:0809.2596; M. Torabian and H.-U. Yee, J. High Energy Phys. 08 (2009) 020.

[5] H. -U. Yee, J. High Energy Phys. 11 (2009) 085; A. Rebhan, A. Schmitt, and S. Stricker, J. High Energy Phys. 01 (2010) 026.

[6] D. T. Son and P. Surowka, arXiv:0906.5044.

[7] M. Lublinsky and I. Zahed, Phys. Lett. B **684**, 119 (2010).

[8] A. V. Sadofyev, V. I. Shevchenko, and V. I. Zakharov, Phys. Rev. D **83**, 105025 (2011); V. P. Kirilin, A. V. Sadofyev, and V. I. Zakharov, arXiv:1203.6312; S. Dubovsky, L. Hui, A. Nicolis, and D. T. Son, arXiv:1107.0731; S. Lin, Phys. Rev. D **85**, 045015 (2012).

[9] V. P. Nair, R. Ray, and S. Roy, arXiv:1112.4022.

[10] M. A. Nowak, M. Rho, and I. Zahed, *Chiral Nuclear Dynamics* (World Scientific, Singapore, 1996).

[11] D. T. Son and N. Yamamoto, arXiv:1203.2697.

[12] A. Burkov amd L. Balents, Phys. Rev. Lett. **107**, 127205 (2011).

[13] K. Sasaki, arXiv:0106190.

[14] A. S. Gorsky, JETP Lett. **48**, 121 (1988).

[15] G. E. Volovik, *The Universe in a Helium Droplet* (Oxford University, New York, 2003).

[16] D. Kharzeev, Phys. Lett. B **633**, 260 (2006); D. E. Kharzeev, L. D. McLerran and H. J. Warringa, Nucl. Phys. **A803**, 227 (2008); D. E. Kharzeev and H. -U. Yee, Phys. Rev. D **83**, 085007 (2011).

[17] K. Landsteiner, E. Megias and F. Pena-Benitez, Phys. Rev. Lett. **107**, 021601 (2011).

[18] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).

[19] X. L. Qi and S. C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).

[20] S. L. Adler and W. A. Bardeen, Phys. Rev. **182**, 1517 (1969); H. Itoyama and A. H. Mueller, Nucl. Phys. **B218**, 349 (1983).

[21] G. V. Dunne and C. A. Trugenberger, Ann. Phys. (N.Y.) **195**, 356 (1989).

[22] A. Vilenkin, Phys. Rev. D **21**, 2260 (1980).

[23] A. Y. Alekseev, V. V. Cheianov and J. Frohlich, Phys. Rev. Lett. **81**, 3503 (1998).

[24] D. Kharzeev and A. Zhitnitsky, Nucl. Phys. **A797**, 67 (2007).