$\mathbb{PT}$-Symmetric Quantum Liouvillean Dynamics

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We discuss a combination of unitary and antiunitary symmetry of quantum Liouvillean dynamics, in the context of open quantum systems, which implies a D_2 symmetry of the complex Liouvillean spectrum. For sufficiently weak system-bath coupling, it implies a uniform decay rate for *all* coherences, i.e., off-diagonal elements of the system's density matrix taken in the eigenbasis of the Hamiltonian. As an example, we discuss symmetrically boundary driven open XXZ spin $1/2$ chains.

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Introduction.—The so-called PT symmetry, a product of a unitary and an antiunitary transformation both of which square to identity, of non-Hermitian Hamiltonians has been introduced by Bender [1] to study the spectral theory of Schrödinger-like operators and nonstandard formulations of quantum mechanics. The generic possibilities [1–6] of having non-Hermitian operators with purely real spectra have recently found impressive experimental applications in nonlinear optics [7,8] and even in LRC electric circuits [9] where PT symmetry is achieved by a subtle combination of active elements, with the gain and the loss distributed in a symmetric way.

In this Letter, we show that the concept of PT symmetry can be introduced, in contrast to Ref. [1], in the context of standard, orthodox quantum mechanics when one considers open, dissipative systems. We discuss a general situation in which a Liouvillean superoperator possesses a combination of unitary and antiunitary master symmetries (transformations in the linear space of generators of Hermiticity preserving dynamical semigroups). We show that, as a consequence of master symmetry, the spectrum of decay rates should have a dihedral (D_2) symmetry in the complex plane. Furthermore, for sufficiently weak system-bath coupling the spectrum has the shape of a cross, with one leg on the real axis, corresponding to dynamics of populations (diagonal operators) in the energy eigenbasis, and the other leg parallel to the imaginary axis, corresponding to the decay of coherences (off-diagonal operators), remarkably, all with the same asymptotic rate. This phenomenon offers a fundamentally new way of controlling decoherence as one deals with a single damping factor. Therefore, our general result should be of interest for a variety of fields which use the methods of open quantum systems [10], ranging from nonequilibrium statistical mechanics, quantum optics, quantum information, and quantum measurement theory, to condensed matter, high-energy theory, and quantum cosmology.

As an example, we demonstrate how our symmetry is realized in the symmetric boundary driven XXZ spin $1/2$ chain [11–14] which is described in terms of the standard Lindblad equation (with a Hermitian Hamiltonian) and the canonical formalism of Markovian open quantum systems. We stress that our considerations assume only local-and-homogeneous-in-time open quantum system's theory, without any need for non-Hermitian central system's Hamiltonians. Yet, the results we find formally faithfully generalize the mathematical framework of PT-symmetric quantum mechanics [1] to a Liouvillean setting.

Quantum dynamics and Liouville space formalism.—Let us consider a system defined on a *finite* Hilbert space $\mathcal{H}$ of N states, with a canonical orthonormal basis $|j\rangle$, $j = 1\ldots N$. Let $\mathcal{B}(\mathcal{H})$ denote the vector space of linear operators over $\mathcal{H}$. N^2 basis states of $\mathcal{B}(\mathcal{H})$ shall be denoted by $E_{j,k} = |j\rangle\langle k|$, $j, k = 1\ldots N$. Introducing the Hilbert-Schmidt inner product (see, e.g., [15]) $(\sigma, \rho) := \mathrm{tr}(\sigma^\dagger \rho)$, $\mathcal{B}(\mathcal{H})$ becomes a Hilbert space, with $\{E_{j,k}\}$ being its orthonormal basis. As arbitrary physical states are elements of $\mathcal{B}(\mathcal{H})$ in the sense of density operators, the generator $\hat{\mathcal{L}}$ of quantum Liouvillean dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \hat{\mathcal{L}}\rho(t) \tag{1}$$

can be considered as an element of $\mathcal{B}(\mathcal{B}(\mathcal{H}))$ and can be, say, in a basis $\{E_{j,k}\}$, represented by the $N^2 \times N^2$ matrix. For general Markovian open quantum systems, the Liouvillean $\hat{\mathcal{L}}$ can be always cast to the Lindblad form [10,16,17]

$$\begin{aligned}\hat{\mathcal{L}} &= -\mathrm{i}(\mathrm{ad}H) + \gamma\hat{\mathcal{D}}, \qquad (\mathrm{ad}H)\rho := [H, \rho],\\ \hat{\mathcal{D}}\rho &:= \sum_m 2L_m\rho L_m^\dagger - L_m^\dagger L_m\rho - \rho L_m^\dagger L_m,\end{aligned} \tag{2}$$

separating the unitary part generated by a *Hermitian* Hamiltonian $H \in \mathcal{B}(\mathcal{H})$, and the dissipator $\hat{\mathcal{D}} \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$, where $\{L_m; m = 1\ldots M\} \subset \mathcal{B}(\mathcal{H})$ is a set of $M \le N^2 - 1$ Lindblad operators, which together with the system-bath *coupling strength* $\gamma > 0$ contain all information that is left about the reservoirs (i.e., environment degrees of freedom) and coupling to them. One can always adjust γ such as to fix the trace of the dissipator $\mathrm{Tr}\hat{\mathcal{D}} = \sum_{j,k}(E_{j,k}, \hat{\mathcal{D}}E_{j,k}) = -\mathrm{Tr}\hat{1} = -N^2$. We note that the abstract part of our discussion will not require a particular Lindblad form (2) of the Liouvillean $\hat{\mathcal{L}}$, but we require only the existence of the following concepts.

Liouvillean $\mathbb{P}$ *transformation.*—Let $\hat{\mathcal{P}} \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ be some *unitary* transformation, $(\hat{\mathcal{P}}\sigma, \hat{\mathcal{P}}\rho) = (\sigma, \rho)$, $\forall\ \sigma, \rho \in \mathcal{B}(\mathcal{H})$, squaring to identity $\hat{\mathcal{P}}^2 = \hat{1}$. Noting that $\hat{\mathcal{P}}^{-1} = \hat{\mathcal{P}}$, we define $\mathbb{P}$ transformation of the Liouvillean as

$$\mathbb{P}\,\hat{\mathcal{L}} := \hat{\mathcal{P}}\,\hat{\mathcal{L}}\,\hat{\mathcal{P}}. \tag{3}$$

Liouvillean $\mathbb{T}$ *transformation.*—The Hilbert-Schmidt inner product completely defines the Hermitian adjoint of the Liouvillean $\hat{\mathcal{L}}^\dagger$, namely, $(\sigma, \hat{\mathcal{L}}^\dagger\rho) = (\hat{\mathcal{L}}\sigma, \rho) = \overline{(\rho, \hat{\mathcal{L}}\sigma)}$, $\forall\ \sigma, \rho \in \mathcal{B}(\mathcal{H})$. We define the $\mathbb{T}$ transformation as

$$\mathbb{T}\,\hat{\mathcal{L}} := \hat{\mathcal{L}}^\dagger. \tag{4}$$

Note that the maps $\mathbb{P}$ and $\mathbb{T}$ can be considered as elements of $\mathcal{B}(\mathcal{B}(\mathcal{B}(\mathcal{H})))$. By introducing the super-Hilbert-Schmidt inner product in $\mathcal{B}(\mathcal{B}(\mathcal{H}))$, $((\hat{\mathcal{X}}, \hat{\mathcal{Y}})) := \mathrm{Tr}(\hat{\mathcal{X}}^\dagger \hat{\mathcal{Y}})$, it can be verified straightforwardly that the map $\mathbb{P}$ is unitary, while the map $\mathbb{T}$ is antiunitary, namely,

$$((\mathbb{P}\hat{\mathcal{X}}, \mathbb{P}\hat{\mathcal{Y}})) = ((\hat{\mathcal{X}}, \hat{\mathcal{Y}})), \tag{5}$$

$$((\mathbb{T}\hat{\mathcal{X}}, \mathbb{T}\hat{\mathcal{Y}})) = ((\hat{\mathcal{Y}}, \hat{\mathcal{X}})), \qquad \forall\ \hat{\mathcal{X}}, \hat{\mathcal{Y}} \in \mathcal{B}(\mathcal{B}(\mathcal{H})). \tag{6}$$

$\mathbb{P}\mathbb{T}$*-symmetric Liouvillean.*—Let $\hat{\mathcal{L}}'$ denote the *traceless* part of the Liouvillean, defined as

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}' - \gamma\hat{1}, \quad \text{with } \gamma := -\mathrm{Tr}\hat{\mathcal{L}}/\mathrm{Tr}\hat{1}. \tag{7}$$

We shall define quantum Liouvillean dynamics as $\mathbb{P}\mathbb{T}$-symmetric if the following key identity holds:

$$\mathbb{P}\,\mathbb{T}\hat{\mathcal{L}}' = -\hat{\mathcal{L}}', \tag{8}$$

or, equivalently, $(\hat{\mathcal{L}}')^\dagger = -\hat{\mathcal{P}}\hat{\mathcal{L}}'\hat{\mathcal{P}}$. This $\mathbb{P}\mathbb{T}$ symmetry can be immediately applied to invert the Liouvillean propagator of generally dissipative dynamics $\hat{\mathcal{U}}(t) := \exp(t\hat{\mathcal{L}})$:

$$\hat{\mathcal{U}}(-t) = e^{2\gamma t}\hat{\mathcal{P}}[\hat{\mathcal{U}}(t)]^\dagger\hat{\mathcal{P}}. \tag{9}$$

Now we are in position to show the following result.

Theorem.—The spectrum of $\mathbb{P}\mathbb{T}$-symmetric Liouvillean (8) has a dihedral group (D_2) symmetry in the complex plane, with the lines of symmetry $\ell_{\mathrm{v}} = -\gamma + \mathrm{i}\mathbb{R}$ and $\ell_{\mathrm{h}} = \mathbb{R}$. More specifically, writing the Liouvillean spectral decompositions

$$\hat{\mathcal{L}}u_\alpha = \lambda_\alpha u_\alpha, \qquad \hat{\mathcal{L}}^\dagger v_\alpha = \bar{\lambda}_\alpha v_\alpha \tag{10}$$

with right and left eigenvectors which can be chosen bi-orthonormal, $(u_\alpha, v_\beta) = \delta_{\alpha,\beta}$, we state that for each eigenvalue of $\hat{\mathcal{L}}'$, $\lambda'_\alpha \equiv \lambda_\alpha + \gamma$, there exist the eigenvalues $\lambda'_\beta = -\bar{\lambda}'_\alpha$ (image across ℓ_{v}) and $\lambda'_\eta = \bar{\lambda}'_\alpha$ (image across ℓ_{h}) with the right and left eigenvectors related via

$$u_\beta = \hat{\mathcal{P}}v_\alpha, \qquad v_\beta = \hat{\mathcal{P}}u_\alpha, \tag{11}$$

$$u_\eta = u_\alpha^\dagger, \qquad v_\eta = v_\alpha^\dagger. \tag{12}$$

In the case of a degenerate eigenvalue λ_α, the eigenvectors $u_{\beta/\eta}$ and $v_{\beta/\eta}$ are to be understood as appropriate members of the right and left eigenspaces, respectively.

Proof.—The ℓ_{v}-reflection spectral symmetry with (11) is a direct consequence of $\mathbb{P}\mathbb{T}$ symmetry (8), after applying it to the left-hand sides of Eqs. (10) and multiplying the resulting equation by $\hat{\mathcal{P}}$. The ℓ_{h}-reflection symmetry with (12), on the other hand, is an immediate consequence of Hermiticity preservation of Liouvillean quantum dynamics, $\hat{\mathcal{L}}(\rho^\dagger) = (\hat{\mathcal{L}}\rho)^\dagger$, which clearly holds for the Lindbladian (2) but also for any other (possibly non-Markovian) meaningful quantum dynamics [17]. ∎

Remarks.—One of the most interesting objects of open quantum dynamics is the steady state $\rho(t \to \infty) = u_1$, corresponding to eigenvalue $\lambda_1 = 0$, which always exists, due to trace preservation in $\mathcal{H}$, which can be expressed as $\hat{\mathcal{L}}^\dagger v_1 = 0$ with $v_1 = \mathbb{1} = \sum_{j=1}^N |j\rangle\langle j|$. This means that Eq. (11) yields already a nontrivial result, namely, that the fastest decaying mode $\lambda_{N^2} = -2\gamma$ is $u_{N^2} = \hat{\mathcal{P}}(\mathbb{1})$.

Even more remarkable observation is that for sufficiently small coupling γ, say, below some critical value $\gamma < \gamma_{\mathrm{PT}}$, the spectrum of decay modes lies strictly on the *cross*, $\{\lambda_\alpha\} \subset \ell_{\mathrm{v}} \cup \ell_{\mathrm{h}}$. First, we show that if an eigenvalue $\lambda_\alpha(\gamma)$ lies on the real line ℓ_{h}, it remains on ℓ_{h} as long as the eigenvalue is *isolated*. First-order nondegenerate perturbation theory tells us that $\mathrm{d}\lambda_\alpha/\mathrm{d}\gamma = (v_\alpha, \hat{\mathcal{D}}u_\alpha)$, whence Hermiticity conservation $\hat{\mathcal{D}}(u_\alpha)^\dagger = \hat{\mathcal{D}}(u_\alpha^\dagger)$, and $u_\alpha^\dagger = u_\alpha$, $v_\alpha^\dagger = v_\alpha$ following from (12) since $\bar{\lambda}_\alpha = \lambda_\alpha$, implies $\mathrm{d}\bar{\lambda}_\alpha/\mathrm{d}\gamma = \mathrm{d}\lambda_\alpha/\mathrm{d}\gamma \in \mathbb{R}$. Second, we show similarly that for an isolated eigenvalue $\lambda_\alpha(\gamma)$ initially on ℓ_{v}, i.e., $\lambda'_\alpha \in \mathrm{i}\mathbb{R}$, we have $\mathrm{d}\lambda'_\alpha/\mathrm{d}\gamma \in \mathrm{i}\mathbb{R}$ as a simple consequence of the $\mathbb{P}\mathbb{T}$ symmetry of the traceless part of the dissipator $\hat{\mathcal{D}}' = \hat{\mathcal{D}} + \hat{1}$, $\mathbb{P}\mathbb{T}\hat{\mathcal{D}}' = -\hat{\mathcal{D}}'$. Namely, from $\bar{\lambda}'_\alpha = -\lambda'_\alpha$ and (11) follows that $u_\alpha = \hat{\mathcal{P}}v_\alpha$, so $\mathrm{d}\bar{\lambda}'_\alpha/\mathrm{d}\gamma = (\hat{\mathcal{D}}'u_\alpha, v_\alpha) = -(u_\alpha, \hat{\mathcal{P}}\hat{\mathcal{D}}'\hat{\mathcal{P}}v_\alpha) = -(v_\alpha, \hat{\mathcal{D}}'u_\alpha) = -\mathrm{d}\lambda'_\alpha/\mathrm{d}\gamma$. The spectrum can then leave the cross $\ell_{\mathrm{v}} \cup \ell_{\mathrm{h}}$, when at some $\gamma = \gamma_{\mathrm{PT}}$ a pair of eigenvalues collides and shoots off into the complex plane. By observing just the motion of the spectral points on the vertical leg ℓ_{v} (while the argument should be quite similar for ℓ_{h}), the critical coupling strength where the $\mathbb{P}\mathbb{T}$ symmetry of the spectrum is spontaneously broken can be estimated heuristically as

$$\gamma_{\mathrm{PT}} \sim \|\hat{\mathcal{D}}'\|^{-1} d^{-2}. \tag{13}$$

Here $\|\hat{\mathcal{D}}'\|$ is the operator norm of the dissipator which estimates the maximal velocities $|\mathrm{d}\lambda_\alpha/\mathrm{d}\gamma|$ and d denotes a typical density of states of H, d^2 giving a typical density of energy differences $\epsilon_j - \epsilon_k$.

What remains to be shown to prove this picture is that initially, for (infinitesimally) small γ, the eigenvalues λ'_α

indeed start on $\mathbb{R}\cup i\mathbb{R}$. This is again easy to demonstrate, working in the eigenbasis of the Hamiltonian H, $H|\psi_j\rangle = \epsilon_j|\psi_j\rangle$, $j=1\ldots N$. For $\gamma=0$, $\lambda'_\alpha = i(\epsilon_j-\epsilon_k)$, where α now labels all N^2 pairs of j, k. Let us assume that the energy spectrum ϵ_j is nondegenerate; i.e., all ϵ_j are different. Then we have exactly N diagonal eigenoperators $d_j = |\psi_j\rangle\langle\psi_j| \equiv u_\alpha = v_\alpha$ for which $\lambda'_\alpha = 0$. In order to understand their motion as we switch on γ, we have to solve the first-order-degenerate-perturbation problem; i.e., we have to diagonalize an $N\times N$ matrix $V_{j,k} := (d_j, \hat{\mathcal{D}}' d_k)$, $Va_\alpha = \xi_\alpha a_\alpha$, where each eigenvalue ξ_α determines the motion of some λ'_α, $d\lambda'_\alpha/d\gamma|_{\gamma=0} = \xi_\alpha$, and the corresponding eigenvector determines the hybridization, $u_\alpha = v_\alpha = \sum a_{\alpha,j} d_j$. The key observation now is that the matrix V is *real* and *symmetric*:

$$\begin{aligned}V_{j,k} &= (d_j, \hat{\mathcal{D}}' d_k) = \langle\psi_j|\hat{\mathcal{D}}'(|\psi_k\rangle\langle\psi_k|)|\psi_j\rangle \\ &= 1 + \sum_m (2|\langle\psi_j|L_m|\psi_k\rangle|^2 - \langle\psi_j|L_m^\dagger L_m|\psi_j\rangle \\ &\quad - \langle\psi_k|L_m^\dagger L_m|\psi_k\rangle) \\ &= V_{k,j} = \bar{V}_{k,j},\end{aligned} \tag{14}$$

proving that these N eigenvalues $\lambda'_\alpha(\gamma)$ remain on the real line. Asymptotically, for small γ the states $u_\alpha(\gamma)$ are diagonal and correspond to *populations* in the energy eigenbasis. The other N^2-N eigenvalues $\lambda'_\alpha(\gamma)$, for $\epsilon_j\neq\epsilon_k$, move on the imaginary line, as shown in the previous paragraph, provided that they are isolated initially for $\gamma=0$, i.e., provided that all the energy spacings $\epsilon_j-\epsilon_k$ are different (nondegenerate). These eigenvalues, asymptotically for small γ, correspond to off-diagonal operators—*coherences*—in the energy eigenbasis, and *all* decay with exactly the same rate $\mathrm{Re}\lambda_\alpha \equiv -\gamma$. This is remarkable and should have experimentally observable consequences; e.g., one should be able to control the decoherence in such a system by handling a single damping factor $e^{-\gamma t}$.

The above scenario which predicts the full Liouvillean spectrum of decay modes to belong to the cross $\ell_v\cup\ell_h$ for some nonempty coupling strength interval $0\le\gamma\le\gamma_{PT}$ is strictly justified only if the two conditions are met, namely, that both the energy spectrum $\{\epsilon_j\}$ and the energy difference spectrum $\{\epsilon_j-\epsilon_k; j\neq k\}$ are nondegenerate. In the nongeneric case when we have a degeneracy in either of the two, it can happen (in the absence of additional selection rules) that already infinitesimal dissipation γ moves the corresponding Liouvillean eigenvalues out into the complex plane, so the spontaneous $\mathbb{PT}$-symmetry breaking of the Liouvillean spectrum may then occur already for a vanishing perturbation $\gamma_{PT}=0$.

Example.—We close by demonstrating our constructions in an interesting example; namely, we consider an open *XXZ* chain of n spins 1/2 with the Hamiltonian

$$H = \sum_{j=1}^{n-1}(2\sigma_j^+\sigma_{j+1}^- + 2\sigma_j^-\sigma_{j+1}^+ + \Delta\sigma_j^z\sigma_{j+1}^z), \tag{15}$$

where $\sigma_j^\pm = \frac{1}{2}(\sigma_j^x \pm i\sigma_j^y)$, σ_j^z, $j=1\ldots n$, are Pauli operators on a product space $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$, with symmetric Lindblad driving acting on the edges of the chain only:

$$L_{1,2} = \tfrac{1}{2}\sqrt{1\pm\mu}\,\sigma_1^\pm, \qquad L_{3,4} = \tfrac{1}{2}\sqrt{1\mp\mu}\,\sigma_n^\pm, \tag{16}$$

with $M=4$, where $\mu\in[-1,1]$ is a *driving parameter* determining the magnetization bias between the left and the right baths. We have here $N=2^n$. This model has been intensively studied recently [11–14,18], and it has been shown to admit exact solutions [14] in the limiting cases of small γ or $\mu=1$ and can exhibit diffusive spin transport (for $|\Delta|>1$) in the linear response regime [13], of small μ and nonsmall $\gamma\sim1$. In order to disclose explicitly the $\mathbb{PT}$ symmetry of such a symmetrically boundary driven *XXZ* chain, it is instructive to identify the operator space $\mathcal{B}(\mathcal{H})$ with the tensor product $\mathcal{H}\otimes\mathcal{H}$, via the isomorphism

$$|\psi\rangle\langle\phi| \leftrightarrow |\psi\rangle\otimes S|\phi\rangle, \tag{17}$$

where $S := \prod_{j=1}^n \sigma_j^x$ is a global spin-flip operation in σ_j^z eigenbasis. Then the Liouvillean is represented as

$$\begin{aligned}\hat{\mathcal{L}} \leftrightarrow\ & \mathbb{1}\otimes\Big(iH - \frac{\gamma\mu}{4}(\sigma_1^z-\sigma_n^z)\Big) - \Big(iH - \frac{\gamma\mu}{4}(\sigma_1^z-\sigma_n^z)\Big)\otimes \\ & \mathbb{1} + \frac{\gamma(1+\mu)}{2}(\sigma_1^+\otimes\sigma_1^- + \sigma_n^-\otimes\sigma_n^+) + \frac{\gamma(1-\mu)}{2} \\ & \times(\sigma_1^-\otimes\sigma_1^+ + \sigma_n^+\otimes\sigma_n^-) - \gamma\mathbb{1}\otimes\mathbb{1}.\end{aligned} \tag{18}$$

Let us define the parity transformation $\hat{\mathcal{P}}$, such that it corresponds to the following operator:

$$\hat{\mathcal{P}} \leftrightarrow \Big(R\prod_{j=1}^n \sigma_j^z\Big)\otimes R, \tag{19}$$

where R is a reflection permutation which reverses the order of sites $j\leftrightarrow n+1-j$; i.e., in the common eigenbasis of σ_j^z, $j=1\ldots n$, it reads $R = \sum_{m_1\ldots m_n\in\{+1,-1\}} |m_n\ldots m_1\rangle\langle m_1\ldots m_n|$. It takes a straightforward calculation to show that indeed $\hat{\mathcal{P}}^2 = \hat{1} \leftrightarrow \mathbb{1}\otimes\mathbb{1}$, and $(\hat{\mathcal{L}}+\gamma\hat{1})^\dagger = -\hat{\mathcal{P}}(\hat{\mathcal{L}}+\gamma\hat{1})\hat{\mathcal{P}}$. In fact, the $\mathbb{PT}$ symmetry (8) holds for each of the three rows of the expression (18) separately. In Fig. 1, we show three different Liouvillean spectra for different values of the coupling constant, before and after the transition $\gamma=\gamma_{PT}$. Numerical experiments indicate that the critical value in the leading order decays exponentially with the chain length, $\gamma_{PT}\propto d^{-2}\propto 4^{-n}$, as estimated in (13). We note that, even for $\gamma\gg\gamma_{PT}$, a substantial fraction of spectral points remain on the line $\ell_v = -\gamma + i\mathbb{R}$; hence, a significant spectral weight for uniform relaxation $e^{-\gamma t}$ is expected for typical (off-diagonal) observables. Remarkably, an important operator in the transport theory, the spin current operator $J = i\sum_{j=1}^{n-1}(\sigma_j^+\sigma_{j+1}^- - \sigma_j^-\sigma_{j+1}^+)$, has vanishing diagonal matrix elements in the energy eigenbasis, so its expectation value decays with a uniform rate $|\mathrm{tr}[\rho(t)J]| \sim e^{-\gamma t}$, for $\gamma<\gamma_{PT}$.

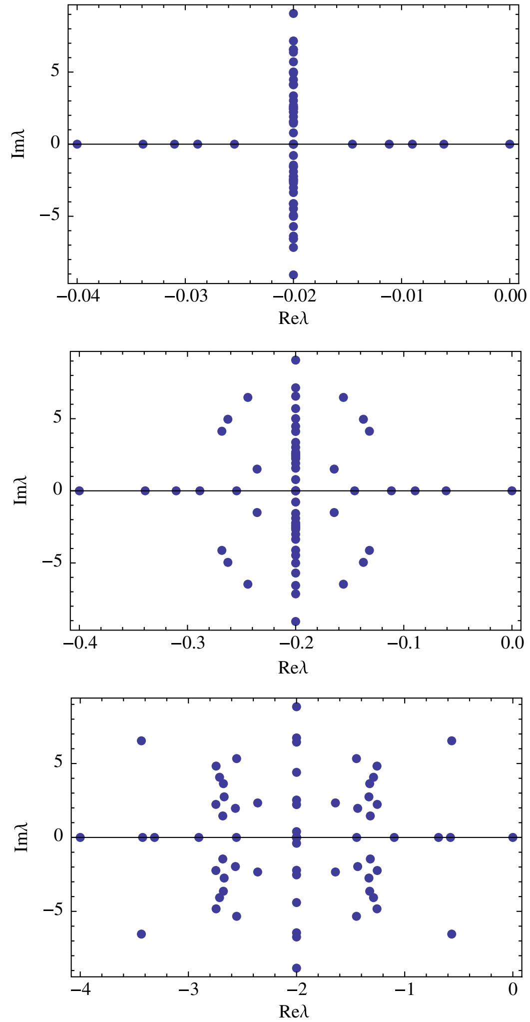

FIG. 1 (color online). The Liouvillean spectrum $\{\lambda_k\}$ for an open *XXZ* chain with $\Delta = 1/2$, $n = 4$, maximum driving $\mu = 1$, and $\gamma = 0.02 < \gamma_{\mathrm{PT}}$ (top), $\gamma = 0.2 > \gamma_{\mathrm{PT}}$ (middle), and $\gamma = 2 > \gamma_{\mathrm{PT}}$ (bottom). Note the dihedral symmetry of the spectrum with the horizontal and vertical symmetry lines, $\ell_{\mathrm{h}} = \mathbb{R}$ and $\ell_{\mathrm{v}} = -\gamma + \mathrm{i}\mathbb{R}$, respectively. Because the *XXZ* chain has a global conservation law $M^z = \sum_j \sigma_j^z$, we consider only the most relevant sector with zero total magnetization $M^z = 0$.

Discussion.—We outlined a general framework for analysis of a combined unitary ($\mathbb{P}$) and antiunitary ($\mathbb{T}$) master symmetry of the most general types of quantum master equations which are local in time. We stress that our analysis remains strictly in the framework of canonical quantum mechanics, so we need no active elements or non-Hermitian system's Hamiltonians for our constructions. $\mathbb{PT}$ symmetry of dissipative Liouvillean dynamics can thus occur only with respect to a shift parallel to the imaginary line which represents an average damping rate. In the asymptotic regime of weak system-bath coupling, the Liouvillean spectrum can be strictly separated with the coherences—the off-diagonal matrix elements of the state—in the energy eigenbasis decaying with a strictly uniform rate. We discussed a simple explicit example of $\mathbb{PT}$-symmetric Liouvillean dynamics in open *XXZ* spin chains, but other interesting and experimentally accessible realizations are possible. For example, the recently studied symmetrically driven Fermi Hubbard chain [12,19] is $\mathbb{PT}$-symmetric as well (as can be easily seen in spin-ladder formulation), but applications to driven open bosonic cold atom systems should also be possible.

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