Disentanglement versus Decoherence of Two Qubits in Thermal Noise

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We show that the influence of thermal noise, simulated by a 2D ferromagnetic Ising spin lattice on a pair of noninteracting, initially entangled qubits, represented by quantum spins, leads to unexpected evolution of quantum correlations. The high temperature noise leads to ultraslow decay of the quantum correlations. Decreasing the noise temperature we observe a decrease of the characteristic decay time scale. When the noise originates from a critical state, a revival of the quantum correlations is observed. This revival becomes oscillatory with a slowly decaying amplitude when the temperature is decreased below the critical region, leading to persistence of the quantum correlations.

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Introduction.—The control of the decay of quantum correlations in simple systems under the influence of external noise is of great importance for the development of quantum information processing devices [1–6]. A prototype open system allowing for analytic description and deeper understanding of the environmental influence, consists of two noninteracting qubits exposed in a classical stochastic field. The evolution of the density matrix and the entanglement in such a system have already been studied extensively in [7–15] under the assumptions of pure Markovian [7,8,10,12,13], Ornstein-Uhlenbeck [11], and random telegraph noise [15] for the representation of the stochastic environment. In the two-qubit system, despite its simplicity, very interesting phenomena like entanglement sudden death (ESD) [9–11,15] and entanglement rebirth (ER) [15] have been observed. These are fundamental processes characterizing the response of a quantum system to its environment.

In experimental conditions an entangled system may be exposed to vacuum noise, phase noise, thermal noise, and various classical noises, as well as mixed combinations of noises. In general these noises are non-Markovian and there is no way to define their properties from first principles. In the present Letter, we will make a step in this direction. We will consider a two-qubit system, represented by quantum spins, coupled to a noise originating from an environment at thermal equilibrium. To achieve this we will form a canonical ensemble of effective magnetic fields generated by the magnetization of 2D classical ferromagnetic Ising spins at fixed temperature. The magnetic field ensemble is equivalent to a stochastic process representing an external noise applied to the two entangled, noninteracting qubits. Depending on the value of the environment temperature, we observe a large variability in the response of the qubit system to the thermal noise. In particular, two unexpected phenomena occur: (i) the ultraslow decay of concurrence for noises originating from a hot environment and (ii) the oscillatory revival of entanglement (OER) when the temperature of the environment generating the noise is decreased below the critical temperature of the 2D Ising model. Our analysis opens up the perspective of using temperature as a tool to control the decay of quantum correlations in quantum information devices.

Phase noise.—We explore the decay of quantum correlations in a system of two qubits (labeled 1 and 2) interacting with an external stochastic magnetic field $B(t)$ which possesses thermal fluctuations. One possibility to generate such a field is to consider the following Hamiltonian, describing a composite system consisting of the two qubits and a thermal magnetic environment:

$$H = H_S + H_E + H_I, \qquad H_S = -\frac{1}{4} g\sigma_{z,1} \otimes \sigma_{z,2},$$
$$H_E = H_{Is}[I_1 \otimes I_2], \qquad H_{Is} = -J\sum_{\langle i,j\rangle} s_i s_j,$$
$$H_I = -\frac{1}{2}\mu\left(\sum_{i=1}^{N^2} s_i\right)[\sigma_{z,1} \otimes I_2 + I_1 \otimes \sigma_{z,2}], \qquad (1)$$

where H_S is the two-qubit interaction term, H_E is the Hamiltonian of the thermal environment, H_{Is} is the ferromagnetic ($J > 0$) 2-D Ising Hamiltonian and H_I describes the interaction of the two qubits with the Ising lattice (we set $\hbar = 1$). In Eq. (1) σ_z is the Pauli matrix while in H_{Is} the sum is over pairs of adjacent spins, each pair counted once. Finally, s_i are dichotomous variables representing the classical spins of the Ising lattice ($s_i = \pm 1$, $i = 1, 2, \ldots$) and I_1, I_2 are the identity matrices in the space of spins 1 and 2. Assuming that $g \ll J$, we neglect the interaction between the qubits. Furthermore, we assume that each qubit interacts with a $N \times N$ subset of classical spins of the periodic lattice consisting its environment. In practice such a situation can be realized when the qubits, being in an appropriate distance from the lattice, couple to a large subset of Ising spins. Summing up over the classical spin variables s_i one obtains the reduced qubit-spin bath interaction Hamiltonian $H_{I,\mathrm{eff}}$:

$$H_{I,\text{eff}} = \frac{1}{Z}\sum_{\{s_m\}} e^{-\beta H_{Is}} H_I[\{s_i\}], \tag{2}$$

where $\{s_m\}$ denotes the set of all possible classical spin configurations and $Z = \sum_{\{s_m\}} e^{-\beta H_{Is}}$ is the partition function of the 2-D Ising model, β being the inverse temperature. Equation (2) defines an effective external magnetic field $B_{\text{eff}} = \frac{1}{Z}\sum_{\{s_i\}} e^{-\beta H_{Is}}(\sum_{k=1}^{N^2} s_k)$, representing the influence of the equilibrated Ising lattice on the two qubits. It is useful to express B_{eff} as

$$B_{\text{eff}} = \int_{\Omega_B} dB B\rho(B),$$

$$\rho(B) = \frac{1}{Z}\sum_{\{s_i\}} e^{-\beta H_{Is}} \delta\Big(B - \sum_{k=1}^{N^2} s_k\Big), \tag{3}$$

defining a probability density $\rho(B)$ for the external field felt by the two qubits (Ω_B is the set of all possible B values). Time dependence can now be introduced by assuming that the density $\rho(B)$ is generated by an ergodic stochastic process $B(t)$.

Then $B_{\text{eff}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T dtB(t)$ together with Eq. (2) and the definition of B_{eff} lead to a time dependent Hamiltonian for the interaction of the qubits with the spin environment

$$H_I(t) = -\tfrac{1}{2}\mu B(t)[\sigma_{z,1} \otimes I_2 + I_1 \otimes \sigma_{z,2}], \tag{4}$$

with $H_{I,\text{eff}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T dtH_I(t)$. The stochastic process $B(t)$ can be simulated by an ensemble of random trajectories $\{B(t)\}$ which build up the density $\rho(B)$. Thus, Eq. (4) together with the ensemble $\{B(t)\}$ introduce time evolution in the two-qubit system compatible with the presence of thermal equilibrium in their spin environment. To generate the ensemble $\{B(t)\}$ we apply an ergodic simulation algorithm (here Metropolis) for the production of thermal equilibrium configurations $\{s_m\}$ of the Ising spins at a fixed temperature T. We record configurations only after a long transient necessary to approach equilibrium. For each configuration, indexed by k, we calculate the effective magnetic field $B^{(k)}$ using $B^{(k)} = \sum_{i=1}^{N^2} s_i^{(k)}$. The time ordering of the $B^{(k)}$ values is dictated by the sequence of the Monte Carlo steps (MCs). Since we consider exclusively equilibrium dynamics, our results are insensitive to the choice of the time unit. For example, recording the $B^{(k)}$ values in each Monte Carlo step ($\Delta t = 1$ MCs) or recording them every fifth Monte Carlo step ($\Delta t = 5$ MCs) leads to exactly the same results for the time dependence of quantum correlations in the two-qubit system. However, it is crucial that the time unit used to measure time evolution is independent of the temperature.

Evolution of the density matrix and the concurrence.—The time-dependent density matrix for the two-qubit system described by the Hamiltonian (4) is obtained by the unitary evolution of the initial density matrix (at $t = 0$) after taking ensemble average over the noise field $B(t)$ [8,12].

This standard procedure leads to (see in [16] for details):

$$\rho(\mathbf{t}) = \begin{pmatrix} \rho_{11} & G(\mu)\rho_{12} & G(\mu)\rho_{13} & G(2\mu)\rho_{14} \\ G^*(\mu)\rho_{21} & \rho_{22} & 0 & G(\mu)\rho_{24} \\ G^*(\mu)\rho_{31} & 0 & \rho_{33} & G(\mu)\rho_{34} \\ G^*(2\mu)\rho_{41} & G^*(\mu)\rho_{42} & G^*(\mu)\rho_{43} & \rho_{44} \end{pmatrix}, \tag{5}$$

where $\rho_{ij} \equiv \rho_{ij}(0)$ and $G(\mu) \equiv G(\mu; t)$ is the characteristic function of the stochastic process defined by $\phi(t) \equiv \int_0^t B(t')dt'$. We also set $\rho_{23}(0) = 0$ in order to allow for complete decoherence and disentanglement in the density matrix under a global stochastic field $B(t)$.

We identify two characteristic decoherence functions that determine the two time scales of the process: (1) the robust decoherence function $\Gamma_R = |G(\mu; t)| \equiv |\frac{\rho_{12}(t)}{\rho_{12}}|$ and (2) the fragile decoherence function $\Gamma_F = |G(2\mu; t)| \equiv |\frac{\rho_{14}(t)}{\rho_{14}}|$.

As a quantitative measure of the entanglement rate we use the concurrence function [17] (normalized by its value at $t = 0$):

$$\Gamma_{\text{Ent}} = C(\rho(t)) \equiv \max\Big[0, \sqrt{\lambda_1} - \sum_{i=2}^{4}\sqrt{\lambda_i}\Big],$$

where λ_i are the eigenvalues of

$$\rho\tilde{\rho} = \rho(\sigma_y^A \otimes \sigma_y^B)\rho^*(\sigma_y^A \otimes \sigma_y^B)$$

in descending order. According to the above definitions it holds: $0 \le \Gamma_R, \Gamma_F, \Gamma_{\text{Ent}} \le 1$. When $\Gamma_R, \Gamma_F, \Gamma_{\text{Ent}} = 0$ the two-qubit system has become completely decoherent or disentangled [16].

Numerical results.—We evaluate the quantities Γ_R, Γ_F, Γ_{Ent} at 40 equidistant time-instants. In our simulations we set $\mu = 1$. For the generation of the magnetic field time-series we have used a 2D Ising 60×60 lattice with periodic boundary conditions. In order to reduce the parameters in our analysis, we have also set $N = 60$ for the subset of the $N \times N$ spins which are coupled with the qubits. For this lattice, the maximum value of the magnetic field is $B_{\max} = 3600$ J. The "critical" temperature of such a system is close to $T_c \cong 2.3\frac{J}{k_B}$ or $\beta J \cong 0.434$. Since the Ising lattice is equilibrated, there is no relaxation and, therefore, a characteristic time scale in the system. However, for the numerical integration of the stochastic magnetic field, we have to assign a value to the time interval Δt between successive MCs. To achieve this, we first define a time scale by demanding that $\frac{1}{2}\mu B_{\max} t_f = 1$ [18]. Then we set $\Delta t = \frac{t_f}{M}$ where M is the length of a random trajectory $B(t_i)$ ($i = 1, \ldots, M$) in terms of MCs. To construct the ensemble $\{B(t)\}$ at a given temperature we use exclusively trajectories of the same length. After constructing the ensembles we calculate the evolution of the density matrix and the concurrence function for different temperatures. Then we repeat the calculation by increasing the number M of MCs per trajectory (smaller Δt). We continue this procedure until we obtain results which do not change by a further increase of M. We use as Δt the value which corresponds to the smallest M value necessary to achieve convergence of the calculated observables. It turns out that $M = 500$ ($\Delta t \approx 10^{-6}$) is sufficient for that purpose. The Monte Carlo steps for a single $B(t)$ time series are realized by the Metropolis algorithm, whereas to get statistically independent configurations [new $B(t)$ time series] for the ensemble $\{B(t)\}$ we make use of the Wolff algorithm [19]. The completeness of the ensemble used to simulate the thermal environment is tested through the calculation of the distribution of the magnetization ($M = \sum_{i=1}^{N^2} s_i$) at the corresponding temperature which by definition coincides with $\rho(B)$. Namely, we increase the number of ensemble configurations until a convergence of the magnetization density is achieved. It turns out that 10000 configurations for each temperature are enough to converge. Our initial density is picked up randomly and has the form

$$\rho(\mathbf{0}) = \begin{pmatrix} 0.2546 & 0.1745 & 0.0481 & 0.1751 \\ 0.1745 & 0.3504 & 0 & 0.0680 \\ 0.0481 & 0 & 0.0243 & 0.0855 \\ 0.1751 & 0.0680 & 0.0855 & 0.3708 \end{pmatrix}. \quad (6)$$

In Figs. 1(a)–1(e), we present the plot of $\Gamma_R(t)$, $\Gamma_F(t)$, and $\Gamma_{\rm Ent}(t)$ for noises originating from the 2D Ising lattice at five different temperatures. For the high temperature noise ($\beta J = 0.01$), we observe an exceptionally slow decrease of all three quantities, i.e., an ultraslow decoherence process. The entanglement decay rate is faster than the fragile decoherence rate which is bounded above by the robust one, as expected. As we approach the critical region from the high temperature regime we observe a faster decay of all three quantities [see Fig. 1(b)]. At $t \approx 1.5 \times 10^{-3}$ [18], we observe the appearance of ESD. However, the ordering of the decay rates remains the same as in Fig. 1(a). In Fig. 1(c) we show the results for a temperature within the critical region. The decay rates are even faster and the ESD appears earlier. We also observe a decoherence "death" and a subsequent partial "rebirth" for both fragile and robust decoherence functions. In Fig. 1(d) we use a temperature just below the critical region. Here, the decay rates are even larger for all three quantities. However, we observe repetitive occurrence of decoherence death and subsequent revival for the fragile decoherence function. As a consequence, there exists a time interval for which the fragile decoherence rate is smaller than that of the robust one contrary to expectations. This effect becomes more spectacular in the case of noise corresponding to even

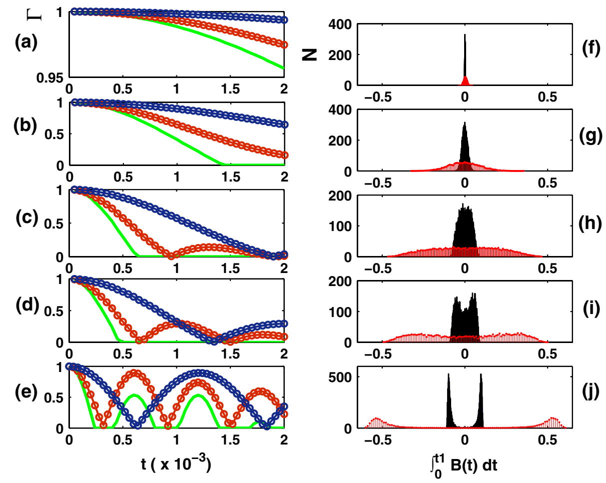

FIG. 1 (color online). (a)–(e) The functions $\Gamma_R(t)$ (blue circles and line), $\Gamma_F(t)$ (red circles and line), and $\Gamma_{\rm Ent}(t)$ (green line) for five different inverse temperature values: (a) $\beta J = 0.01$, (b) $\beta J = 0.41$, (c) $\beta J = 0.436$, (d) $\beta J = 0.443$, and (e) $\beta J = 0.465$. (f)–(j) The corresponding distributions $p(\phi)$ of $\phi(t_1)$ used in the calculation of the decoherence and disentanglement functions (a)–(e) for two different times: $t_1 = 4 \times 10^{-6}$ (black histogram), $t_1 = 2 \times 10^{-5}$ (red histogram).

lower temperatures as demonstrated in Fig. 1(e). Here we observe OER in addition to the periodic decoherence rebirth for the fragile and robust decoherence functions. Notice that the frequency of the periodic fragile decoherence rebirth is approximately twice the corresponding robust one. It is worth noticing that a remarkable property induced by OER is the occasional existence of time intervals for which the disentanglement rate is slower than one of the two decoherence rates.

Amplitude noise.—We consider two similar stochastic local fields $B_1(t)$, $B_2(t)$ acting as an amplitude noise on the qubits $\sigma_{z,1}$ and $\sigma_{z,2}$, respectively. We assume that both fields are effectively generated by the spins of a $N \times N$ 2D Ising lattice at fixed temperature with periodic boundary conditions in the manner described previously. For each temperature value, we generate two different effective magnetic field ensembles $\{B_1(t)\}$ and $\{B_2(t)\}$ performing two independent simulations starting in each case from a different initial configuration of the lattice spins. In such a system the time evolution of the density matrix is determined by the Kraus operators given in [20]. These operators can be parametrized in terms of the function $p(t)$ (the analogue of the characteristic function $G(\mu; t)$ for the phase noise) defined through

$$\frac{dp(t)}{dt} = -\int_0^t d\tau f(t-\tau)p(\tau), \quad (7)$$

where $f(\tau) = \langle B_1(t+\tau)B_1(t)\rangle = \langle B_2(t+\tau)B_2(t)\rangle$ the noise correlation function. Solving numerically Eq. (7) and inserting the result in the expression of the Kraus

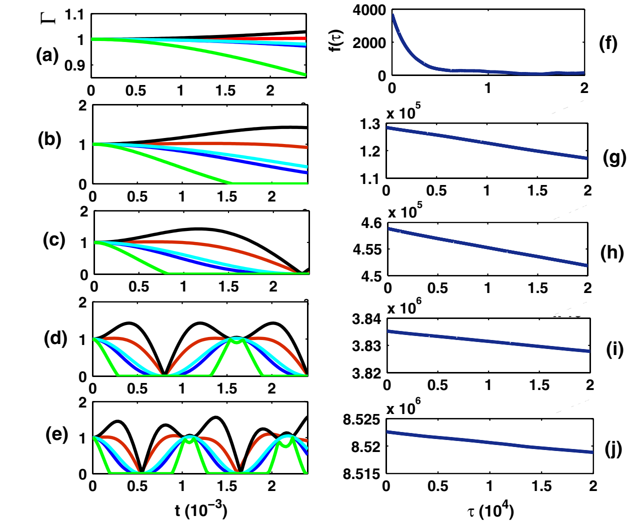

FIG. 2 (color online). (a)–(e) The "decoherence functions" ($\Gamma_{ij}(t) \equiv |\frac{\rho_{ij}(t)}{\rho_{ij}(0)}|$) $\Gamma_{34}(t)$ (black line), $\Gamma_{24}(t)$ (red line), $\Gamma_{14}(t)$ (cyan line), $\Gamma_{12}(t)$ (blue line) and $\Gamma_{Ent}(t)$ (green line) for five different inverse temperature values: (a) $\beta J = 0.01$, (b) $\beta J = 0.41$, (c) $\beta J = 0.436$, (d) $\beta J = 0.443$ and (e) $\beta J = 0.465$. (f)–(j) The corresponding noise correlation functions $f(\tau)$ of the field $B_1(t)$ used in the calculation of the decoherence and disentanglement functions (a)–(e).

operators [20], we obtain for the evolution of the initial density matrix $\rho(0)$ (6) the results shown in Fig. 2.

Similarly to the phase noise case, we observe slow decay at high temperature, sudden death at critical temperature, and OER at low temperature noises. One major difference, typical for amplitude noise [9,10,20], is that some decoherence functions [e.g., $\Gamma_{34}(t)$] obtain values greater than 1.

Discussion and concluding remarks.—It is evident that the characteristic function $G(\mu; t)$ of the stochastic process $\phi(t)$ (determining the decoherence and disentanglement functions in the case of phase noise) tends to zero more rapidly with the increase of the variance of the associated probability density function [8,12,16].

At high temperatures, the distribution of the effective magnetic field felt by the two qubits tends to a very skew Gaussian with a small variance and zero mean, since it is generated by the average magnetization over a macroscopic domain of the Ising lattice. The variance grows with the decrease of temperature. This is the reason why contrary to our intuition for high temperature noise there is a very slow decrease of the decoherence and disentanglement functions which fastens with the decrease of noise temperature. At the critical region and below we can obtain the ESD phenomenon and a gradual revival of the decoherence functions which finally leads us to an oscillatory entanglement rebirth at low temperatures. This phenomenon is attributed to the presence of two maxima in the magnetization distribution due to the spontaneous symmetry breaking of the $Z(2)$ symmetry [Fig. 1(j)]. Approximating this distribution with a sum of two Gaussians with opposite mean values and the same variance, we obtain through Fourier transform an oscillatory characteristic function [16]. Note that at this temperature ($\beta J = 0.465$), the two parts of the probability function are still connected. At lower temperatures, close to $T = 0$, the two parts disconnect and the magnetic field values are distributed only around one of the two maxima. Then the noise distribution is described by a single Gaussian with a tiny variance. Thus, we will obtain results similar to those of high temperatures [Fig. 1(a)] with even slower decay of the correlation functions.

It is remarkable that the classical (noise) correlation functions Figs. 2(f)–2(j) present a quite opposite time dependence, decaying very fast at high temperatures and very slow at low temperatures. These correlation functions can be used in order to explain the effect of the amplitude noise. At high temperatures the correlation function has a relatively small magnitude [Fig. 2(f)] and from Eq. (7) we obtain that $p(t)$ (and consequently also the decoherence and disentanglement functions) is almost constant. On the contrary, the correlations at low temperatures are almost constant [Fig. 2(j)] and thus from (7) we obtain an oscillatory behaviour for $p(t)$.

The presented analysis uses exclusively concurrence to quantify entanglement decay. Currently there is increasing interest for quantum correlations beyond entanglement. An appropriate measure of these correlations is quantum discord [21]. A handy scheme for its calculation, within a class of two-qubit states, is proposed in [22] and it can be directly applied to the system considered here. Preliminary results indicate that the decay of quantum discord depends strongly on the temperature of the thermal environment displaying similar behaviour with the concurrence [16]. However, a more detailed discussion of this subject goes beyond the aim of the present Letter.

In conclusion, we have shown that thermal noise influences significantly the decoherence and disentanglement processes in a two-qubit system. The large variability in the time dependence of the associated decay functions by changing the noise temperature has universal characteristics, suggesting that thermal noise can be a useful tool to control the evolution of quantum correlations in open quantum systems.

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[1] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE Conference on Computers, Systems, and Signal Processing, Bangalore* (IEEE, New York, 1984), p. 175.
[2] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[3] A. Muller, H. Zbinden, and N. Gisin, Europhys. Lett. **33**, 335 (1996).
[4] D. Deutsch and R. Josza, Proc. R. Soc. A **439**, 553 (1992).
[5] P. W. Shor, SIAM J. Comput. **26**, 1484 (1997).

[6] M. A. Nielson and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[7] T. Yu and J. H. Eberly, Phys. Rev. B **66**, 193306 (2002).
[8] T. Yu and J. H. Eberly, Phys. Rev. B **68**, 165322 (2003).
[9] T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004).
[10] T. Yu and J. H. Eberly, Phys. Rev. Lett. **97**, 140403 (2006).
[11] T. Yu and J. H. Eberly, Opt. Commun. **283**, 676 (2010).
[12] K. Ann and G. Jaeger, Phys. Rev. B **75**, 115307 (2007).
[13] M. Ban, S. Kitajima, and F. Shibata, Phys. Lett. A **373**, 3614 (2009).
[14] M. Ban, Eur. Phys. J. D **58**, 415 (2010).
[15] D. Zhou, A. Lang, and R. Joynt, Quantum Inf. Process. **9**, 727 (2010).
[16] A. V. Zampetaki and F. K. Diakonos, ''Disentanglement vs Decoherence of Two Qubits in a General Stochastic Environment'' (unpublished).
[17] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).
[18] All the quantities are measured in the system of units: $J = 1$, $\mu = 1$, $\hbar = 1$, and $k_B = 1$.
[19] M. E. J. Newman and G. T. Barkema, *Monte Carlo Methods in Statistical Physics* (Oxford University Press, New York, 1999).
[20] S-T. Wu, Chin. J. Phys. (Taipei) **50**, 118 (2012).
[21] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).
[22] S. Luo, Phys. Rev. A **77**, 042303 (2008).